REPORT RESUMES

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SEQUENCE CHARACTERISTICS OF TEXT MATERIALS AND TRANSFER OF LEARNING, PART 2 APPENDICES--TASKS AND TESTS.
BY- DELLA-PIANA, GABRIEL M. AND OTHERS
UTAH UNIV., SALT LAKE CITY, BUREAU OF EDUC. RES.
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THIS VOLUME IS AN APPENDIX TO ED 003 346. SAMPLE COPIES OF THE TESTS AND INSTRUCTIONAL MATERIALS USED IN THE STUDIES ARE INCLUDED. DESCRIPTIONS INCLUDE EXPERIMENTS ON DISCOVERY VERSUS EXPOSITORY SEQUENCING IN A PROGRAMED UNIT ON SUMMING NUMBER SERIES. THESE MATERIALS ARE PRESENTED TO PROVIDE CONCERNED PERSONS WITH A BETTER UNDERSTANDING OF THE TREATMENTS USED, AND FOR REPLICATION OR EXTENSION OF THE STUDIES. (RS)

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SEQUENCE CHARACTERISTICS OF TEXT MATERIALS AND TRANSFER OF LEARNING

PART II APPENDICES: TASKS AND TESTS

GABRIEL M. DELLA-PIANA, Project Director

AUTHORS:

BLAINE R. WORTHEN, GARTH M. ELDREDGE, GABRIEL M. DELLA-PIANA

BUREAU OF EDUCATIONAL RESEARCH
UNIVERSITY OF UTAH • SALT LAKE CITY, UTAH 84112

DECEMBER 1965

The research reported herein is Part II of the Final Report of Project 2277 supported under the Cooperative Research Program of the Office of Education, U.S. Department of Health, Education and Welfare.

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Gabriel M. Della-Piana, Project Director

Authors

Blaine R. Worthen
Garth M. Eldredge
Gabriel M. Della-Piana

BUREAU OF EDUCATIONAL RESEARCH
UNIVERSITY OF UTAH
Salt Lake City, Utah 84112

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PREFACE

This volume is Part II of a final report of a project on "Sequence Characteristics of Text Materials and Transfer of Learning." Part I, "Experiments in Discovery Learning," is published as a separate volume reporting procedures and results of the experiments. The present volume includes tests and instructional materials used in the studies. Discovery learning has been variously defined in different studies reported in the literature. Since the materials of instruction essentially define the treatment, they are made available here for the researcher interested in a closer examination for purposes of better understanding, replicating, or extending the studies.

For a discussion of the procedures for developing the materials see Part I of the report. Also, see Part I for acknowledgments to those who have assisted in the conduct of the study.

The Appendices to the Experiment on "Discovery vs. Expository Sequencing in Six Weeks of Classroom Instruction in Mathematics" include materials, instruments and procedures for that study prepared primarily by Blaine R. Worthen in consultation with Stanley M. Jencks.

The Appendices to the Experiment on "Discovery vs. Expository Sequencing in a Programed Unit on Summing Number Series" include the materials for that study prepared primarily by Garth M. Eldredge and Gabriel M. Della-Piana.

Project Director

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APPENDIX I

Materials Used With Treatment D—A Discovery Method

PAGE AND ABBREVIATION KEY FOR EACH UNIT IN TREATMENT D MATERIALS

raD - Real number addition, discovery treatment A1.02

rmD - Real number multiplication, discovery treatment . . A1.12

dpD - Distributive principle of multiplication over addition, discovery treatment A1.31

enD - Exponential notation, discovery treatment A1.52

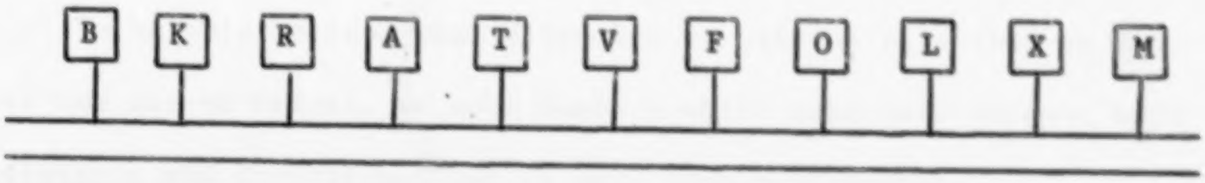
meD - Multiplication of numbers expressed in exponential form, discovery treatment A1.59

deD - Division of numbers expressed in exponential form, discovery treatment A1.64

Materials Used With Treatment D—A Discovery Method

I. Exploration Exercises--raD

A. Imagine a road which has markers placed one mile apart at the side of the road. The markers are labeled with letters. The distance from R to A is one mile; for example:



How far is it from K to T?____________________

Suppose you are at the point F on this road and decide to take a trip of three miles. Where will you be at the end of your trip?

____________________ (Are you sure?)

Suppose you and a friend are at the point T on this road. Each of you decides to take a trip that will take you two miles from T. How many miles apart will you be at the end of your trips?

____________________ Where on the road will you end your trips?

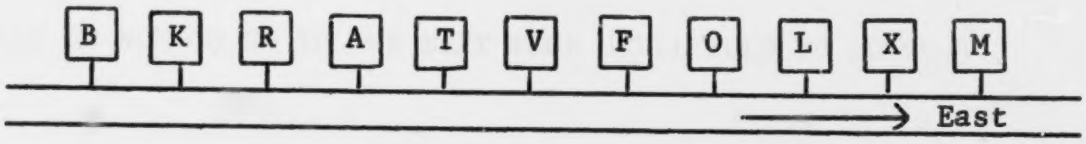
Friend __________; You __________. (Might there be other possibilities?)

It is easy to see that you cannot give a definite single answer to either of the above questions. You must know more than the distance involved in each trip, you must also know something about the direction.

The numbers with which you have been working in school since the first grade can be used in measuring distances for trips along this road. But, these numbers (let's call them numbers of arithmetic) do not tell you in which direction the trips were made. If two people start at T and make two-mile trips, you don't know whether they will be together at the end of their trips or whether they will be four miles apart. Similarly, if you start at F and take a trip of three miles, you may end up at A or you may end up at X.

To be able to know what direction to take on this road as well as how far to travel, we need numbers which take into account both distance and direction. Let us call them real numbers.

The distance of each trip can be measured very well by our normal numbers of arithmetic. But we need to add something to the usual numeral to tell us what direction to take. Let us do this by including an arrow. For example, let us draw the road again, with the addition of an arrow indicating which direction is east. We will use the symbol " $\overset{\rightarrow}{3}$ " (pronounced "right three") to mean a trip of 3 miles to the east. In working with numbers like $\overset{\rightarrow}{4}$ and $\overset{\leftarrow}{6}$, you will want to talk about them as well as write them. So you need to decide upon the pronunciation of their numerals. Let's agree to pronounce " $\overset{\rightarrow}{4}$ " as "right four" and " $\overset{\leftarrow}{6}$ " as "left six."

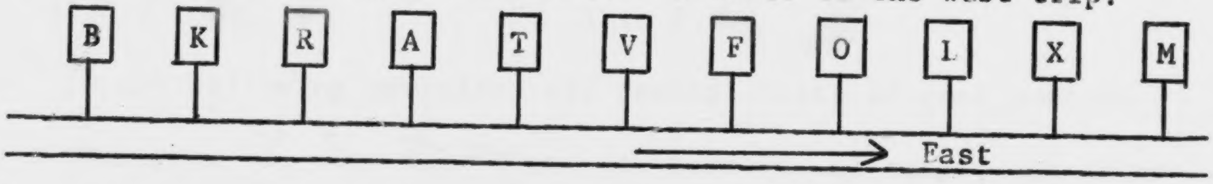


Now, suppose you are at the point F and decide to take a trip of $\overset{\rightarrow}{3}$ miles. Where would you end up? __________ (Since $\overset{\rightarrow}{3}$ would measure a trip of three miles in the direction indicated by the arrow, you would end up three miles east of your starting point, or at X.) Suppose two people start at T and make two-mile trips; your trip being measured by $\overrightarrow{2}$ and your friend's by $\overleftarrow{2}$. Will you be together or 4 miles apart at the end of these trips? ________

(Since $\overrightarrow{2}$ measures a trip of 2 miles to the right, this would take you to F and since $\overleftarrow{2}$ measures a trip of 2 miles to the left, this trip would take your friend to R. Therefore, at the end of these trips, you and your friend would be 4 miles apart.)

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

B. Let us agree that in this exercise the number $\overrightarrow{3}$ measures a 3-miles-to-the-east trip. Then $\overleftarrow{3}$ measures a 3-miles-to-the-west trip.



1. Give the real numbers which measure the trips listed.

Example: K to T

Solution: $\overrightarrow{3}$

a. V to A b. A to V c. A to X

d. A to F e. L to R f. B to R

2. List three trips which are measured by each of the given real numbers.

Example: $\overrightarrow{2}$

Solution: K-A, T-F, O-X

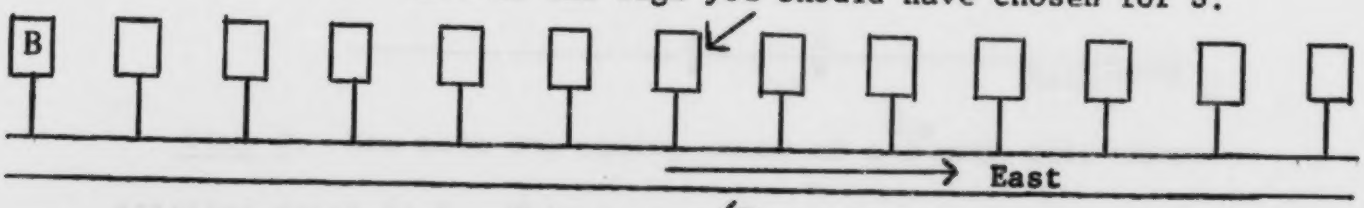
a. $\overleftarrow{3}$ b. $\overleftarrow{1}$ c. $\overrightarrow{8}$

d. $\overleftarrow{5}$ e. $\overrightarrow{4}$ f. $\overrightarrow{6}$

C. Imagine taking trips along an east-west road, and trips to the east are measured by right real numbers. Use the following measurement of trips to fill the markers correctly.

Example: B to S, $\overrightarrow{6}$

Solution: Find the road sign 6 miles east of B and label it S. This is the sign you should have chosen for S.



1. A trip from S to Y is measured by $\overleftarrow{4}$

2. A trip from N to Y is measured by $\overleftarrow{2}$

3. N to Q, $\overleftarrow{3}$ 4. X to R, $\overleftarrow{2}$

5. T to C, $\overleftarrow{2}$ 6. Y to P, $\overrightarrow{5}$ 7. J to H, $\overleftarrow{2}$

8. Q to X, $\overrightarrow{5}$ 9. S to T, $\overrightarrow{5}$ 10. R to C, $\overrightarrow{6}$

11. J to T, $\overrightarrow{1}$ 12. A to R, $\overleftarrow{9}$

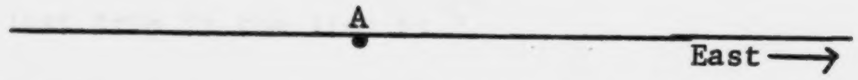
D. In the following exercises are several lists of real numbers. (Example: $\overrightarrow{2}$, $\overleftarrow{3}$, $\overrightarrow{1}$.) These are measures of trips, one after another, along an east-west road.) The first real number in each list measures a trip which starts at a point we shall call A. The second number measures a trip whose starting point is the ending point of the trip taken from A, etc.

You are to tell the location of the ending point of the last trip with respect to A by giving the real number which measures the trip from A to that ending point.

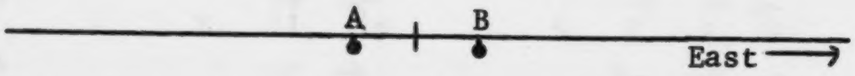
Example 1: $\overrightarrow{2}$, $\overrightarrow{3}$

Solution:

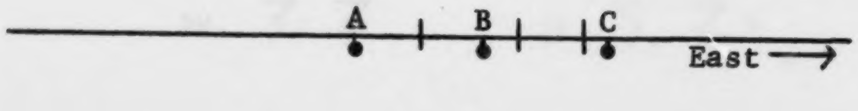
Step 1. One way to solve this problem is to make a sketch of an east-west road and mark a point A.



Step 2. Mark the ending point of the trip measured by $\overrightarrow{2}$. This you can do by marking off units on the road, counting the appropriate number to the right and calling that point B.



Step 3. Now mark the ending point of a $\overrightarrow{3}$ unit trip whose starting point is B. This is a point C, 3 units to the right of B.

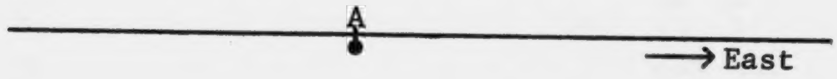


So, C is 5 units to the right of A. This means that the real number which measures the trip from A to the ending point of the last trip in the list is $\overrightarrow{5}$.

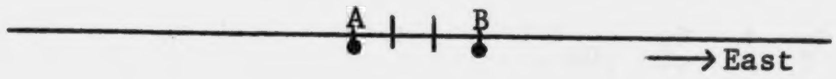
Example 2: $\overrightarrow{3}$, $\overrightarrow{5}$, $\overleftarrow{10}$

Solution:

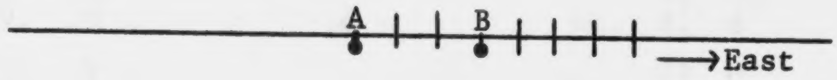
Step 1.



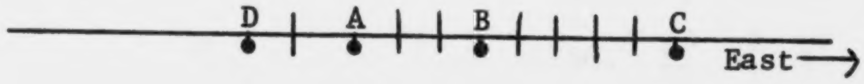
Step 2.



Step 3.



Step 4.



So, the real number which measures the trip from A to the ending point of the last trip in the list is $\overleftarrow{2}$.

Now do the following problems. Use drawings as long as you need to.

1. $\overrightarrow{2}$, $\overrightarrow{7}$
2. $\overleftarrow{3}$, $\overleftarrow{6}$
3. $\overrightarrow{4}$, $\overleftarrow{3}$
4. $\overrightarrow{6}$, $\overleftarrow{8}$
5. $\overleftarrow{12}$, $\overrightarrow{3}$
6. $\overleftarrow{5}$, $\overrightarrow{8}$
7. $\overrightarrow{3}$, $\overrightarrow{7}$, $\overrightarrow{5}$
8. $\overrightarrow{17}$, $\overleftarrow{18}$
9. $\overrightarrow{2}$, $\overrightarrow{9}$, $\overleftarrow{1}$
10. $\overrightarrow{5}$, $\overleftarrow{7}$, $\overrightarrow{8}$
11. $\overrightarrow{7}$, $\overrightarrow{6}$
12, $\overleftarrow{2}$, $\overrightarrow{2}$, $\overrightarrow{2}$
13. $\overrightarrow{7}$, $\overrightarrow{3}$
14. $\overrightarrow{3}$, $\overrightarrow{3}$
15. $\overrightarrow{2}$, $\overrightarrow{12}$
16. $\overrightarrow{4}$, $\overleftarrow{2}$
17. $\overleftarrow{3}$, $\overleftarrow{6}$
18. $\overrightarrow{8}$, $\overrightarrow{1}$
19. $\overrightarrow{10}$, $\overrightarrow{4}$
20. $\overrightarrow{5}$, $\overrightarrow{4}$
21. $\overleftarrow{2}$, $\overrightarrow{4}$

22. $\overleftarrow{4}, \overleftarrow{3}$
23. $\overleftarrow{3}, \overleftarrow{5}$
24. $\overrightarrow{6}, \overrightarrow{10}$
25. $\overleftarrow{12}, \overleftarrow{1}$
26. $\overleftarrow{1}, \overleftarrow{4}$
27. $\overleftarrow{6}, \overleftarrow{8}$
28. $\overrightarrow{5}, \overleftarrow{3}$
29. $\overleftarrow{12}, \overleftarrow{4}$
30. $\overleftarrow{11}, \overleftarrow{11}$

Don't guess at the answers; occasionally check your work by sketching the road. Let's go on!

31. $\overleftarrow{2}, \overleftarrow{1}, \overleftarrow{3}$
32. $\overrightarrow{20}, \overrightarrow{5}$
33. $\overrightarrow{2}, \overrightarrow{2}, \overrightarrow{4}$
34. $\overrightarrow{7}, \overleftarrow{1}$
35. $\overrightarrow{3}, \overrightarrow{8}, \overrightarrow{2}$
36. $\overleftarrow{50}, \overleftarrow{20}$
37. $\overleftarrow{4}, \overleftarrow{3}, \overleftarrow{3}$
38. $\overrightarrow{10}, \overrightarrow{20}, \overrightarrow{10}$
39. $\overleftarrow{4}, \overleftarrow{10}$
40. $\overrightarrow{3}, \overrightarrow{7}, \overrightarrow{5}$
41. $\overleftarrow{5}, \overrightarrow{6}$
42. $\overrightarrow{10}, \overleftarrow{6}$
43. $\overrightarrow{50}, \overrightarrow{100}, \overrightarrow{50}$
44. $\overrightarrow{8}, \overrightarrow{8}$
45. $\overleftarrow{500}, \overleftarrow{500}, \overleftarrow{1}$
46. $\overrightarrow{6}, \overleftarrow{4}$
47. $\overrightarrow{8}, \overleftarrow{6}$
48. $\overrightarrow{7}, \overleftarrow{5}$
49. $\overrightarrow{11}, \overleftarrow{4}$
50. $\overrightarrow{1}, \overrightarrow{1}, \overrightarrow{1}$
51. $\overleftarrow{3}, \overrightarrow{5}$
52. $\overrightarrow{5}, \overleftarrow{3}$
53. $\overleftarrow{4}, \overrightarrow{2}$
54. $\overleftarrow{6}, \overrightarrow{8}$
55. $\overleftarrow{5}, \overrightarrow{7}$
56. $\overrightarrow{11}, \overrightarrow{10}$
57. $\overleftarrow{5}, \overrightarrow{3}$
58. $\overrightarrow{10}, \overleftarrow{9}, \overrightarrow{1}$
59. $\overrightarrow{3}, \overleftarrow{5}$
60. $\overrightarrow{45}, \overleftarrow{5}$
61. $\overleftarrow{1000}, \overrightarrow{1}$
62. $\overleftarrow{5}, \overrightarrow{8}, \overleftarrow{3}$
63. $\overleftarrow{11}, \overleftarrow{1}, \overrightarrow{2}$
64. $\overrightarrow{4}, \overrightarrow{2}, \overrightarrow{14}$

These last problems should really challenge you!

65. $\overrightarrow{10}, \overleftarrow{10}, \overrightarrow{1}$
66. $\overrightarrow{40}, \overrightarrow{5}, \overleftarrow{85}$
67. $\overrightarrow{183}, \overleftarrow{83}$
68. $\overrightarrow{71}, \overleftarrow{71}$
69. $\overrightarrow{4000}, \overleftarrow{4001}$
70. $\overrightarrow{100}, \overrightarrow{100}, \overleftarrow{201}$

E. Numbers which take into account both distance and direction (such as the $\overrightarrow{8}$ notation we used) are used by other people as well. They are called real numbers. Other people use different symbols to name them than we have used so far. It is important that you become familiar with the more standard symbols which are used by mathematicians.

The real numbers which we have called "right numbers" are commonly called "positive numbers." The real numbers which we have called "left numbers" are commonly called "negative numbers."

In naming positive numbers we shall use a " + " instead of the "→." For example, we shall write "+3" instead " $\overrightarrow{3}$." In naming negative numbers we shall use a " -" instead of "←." So, we shall write "-3" instead of " $\overleftarrow{3}$."

F. Each of the following exercises gives the measure of successive trips. Find the real number which would measure the direct trip from the starting point of the first trip to the ending point of the second trip.

Example: $^{+}8$, $^{+}3$

Solution: $^{+}11$

1. $^{-}2$, $^{+}6$
2. $^{-}2$, $^{-}3$
3. $^{+}3$, $^{+}8$
4. $^{+}3$, $^{-}4$
5. $^{-}2$, $^{+}8$
6. $^{-}5$, $^{+}3$
7. $^{-}2$, $^{-}8$
8. $^{-}8$, $^{+}9$
9. $^{+}6$, $^{+}1$
10. $^{-}7$, $^{+}11$
11. $^{+}3$, $^{+}7$
12. $^{+}3$, $^{-}23$
13. $^{+}20$, $^{-}19$
14. $^{-}19$, $^{-}20$
15. $^{+}20$, $^{+}19$
16. $^{-}20$, $^{+}19$

G. The idea of taking two trips, the second being added to the first suggests the notion of "adding" real numbers. This is, in fact, what is meant by addition of real numbers.

Since you are adding real numbers, let us replace the comma with an addition sign and continue to operate in exactly the same fashion as before.

Example 1: $^{+}9$, $^{-}4$ gives a direct trip of $^{+}5$

Therefore, $^{+}9 + {}^{-}4 = {}^{+}5$

Example 2: $^{+}2$, $^{+}8 = {}^{+}10$

$^{+}2 + {}^{+}8 = {}^{+}10$

1. $^{-}2 + {}^{+}3$
2. $^{+}4 + {}^{+}5$
3. $^{-}2 + {}^{+}8$
4. $^{+}4 + {}^{-}3$
5. $^{+}7 + {}^{-}8$
6. $^{+}10 + {}^{+}5$
7. $^{+}9 + {}^{-}3$
8. $^{+}12 + {}^{+}15$
9. $^{+}12 + {}^{+}3$
10. $^{-}2 + {}^{-}5$
11. $^{+}4 + {}^{+}4$
12. $^{+}7 + {}^{-}3$
13. $^{+}7 + {}^{+}3$
14. $^{-}7 + {}^{+}3$
15. $^{-}7 + {}^{-}3$
16. $^{-}2 + {}^{+}2$
17. $^{-}3 + {}^{-}5$
18. $^{-}12 + {}^{+}10$
19. $^{+}2 + {}^{-}1$
20. $^{+}3 + {}^{+}1$
21. $^{-}2 + {}^{+}1$
22. $^{+}6 + {}^{-}7$
23. $^{+}9 + {}^{-}8$
24. $^{-}14 + {}^{+}13$

Here are some that are a little more difficult.

25. $^{+}102 + {}^{-}82$
26. $^{+}750 + {}^{-}350$
27. $^{+}2 + ({}^{-}2 + {}^{+}3)$
28. $^{-}42 + {}^{-}52$
29. $^{+}\frac{1}{2} + {}^{-}\frac{1}{2}$
30. $^{+}190 + {}^{-}81$
31. $^{-}190 + {}^{+}81$
32. $^{+}81 + {}^{-}190$
33. $^{-}7\frac{1}{2} + {}^{+}8\frac{1}{2}$
34. $^{+}3111 + {}^{-}2000$
35. $({}^{-}3 + {}^{+}4) + {}^{+}1$
36. $({}^{+}3 + {}^{-}15) + {}^{+}12$
37. $^{+}2 + ({}^{+}28 + {}^{-}15)$
38. $({}^{-}99 + {}^{-}1) + {}^{+}30$
39. $^{+}87 + ({}^{+}13 + {}^{-}100)$
40. $({}^{-}5 + {}^{+}3) + ({}^{-}8 + {}^{+}1)$

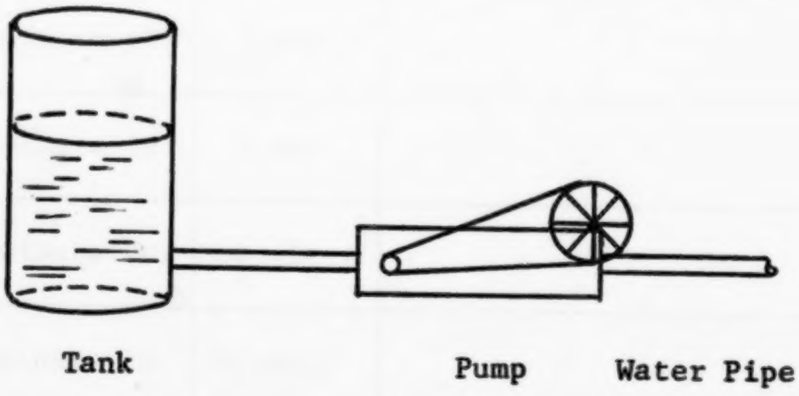
H. **Fill the blanks with numerals which will make the following sentences true.**

1. $^{+}12 + {}^{-}14 = \underline{\qquad}$
2. $\underline{\qquad} + {}^{+}2 = {}^{+}10$
3. $^{-}7 + \underline{\qquad} = {}^{+}2$
4. $^{-}142 + {}^{+}142 + {}^{-}31 = \underline{\qquad}$
5. $\underline{\qquad} + {}^{-}3 = {}^{+}7$
6. $^{+}11 + \underline{\qquad} = 0$
7. $\underline{\qquad} + {}^{-}7 = {}^{-}7$
8. $^{-}981 + {}^{+}981 = \underline{\qquad}$
9. $^{-}37 + {}^{+}672 + \underline{\qquad} = {}^{+}672$
10. $^{+}12 + \underline{\qquad} = {}^{+}12$
11. $^{-}4 + \underline{\qquad} = {}^{+}8$
12. $\underline{\qquad} + {}^{-}2 = {}^{-}4$
13. $^{+}10 + \underline{\qquad} = {}^{+}18$
14. $^{+}70 + {}^{-}20 = \underline{\qquad}$
15. $^{-}3 + \underline{\qquad} = {}^{+}3$
16. $^{+}8 + \underline{\qquad} = {}^{+}8 + {}^{-}971$
17. $^{-}3 + 0 = \underline{\qquad}$
18. $\underline{\qquad} + {}^{+}1 = {}^{-}1$
19. $^{+}87 + \underline{\qquad} = \underline{\qquad} + {}^{+}87$
20. $({}^{+}11 + {}^{-}1) + ({}^{+}1 + {}^{-}11) = \underline{\qquad}$
21. $^{-}1000 + {}^{-}1000 = \underline{\qquad}$
22. $^{+}618 + {}^{-}8 = \underline{\qquad}$

II. Exploration Exercises--rmD

A. Imagine a tank which contains a certain amount of water. The tank is made of glass, thus permitting you to observe the amount of water (water-volume) in the tank.

A pump is attached to the tank by means of a water pipe. Here is a picture of the situation we have imagined thus far.



Suppose that the pump is turned on and that 2 gallons of water flow through the pipe each minute. At the end of 3 minutes, the pump is turned off. What change in the water-volume (amount of water in the tank) do you see? __________ In order to answer this question, you need to know whether the flow of water was into the tank or out of the tank, since the pump can pump in either direction. Suppose that the flow was into the tank. Now what change in the water-volume do you see? __________ (Answer: The water-volume would increase /go up/ 6 gallons.)

Suppose that the flow was out of the tank. What change would you now see in the water-volume? __________ (Answer: The water-volume would decrease /go down/ 6 gallons.)

B. Complete the table.

	Pump	Time	Observed Change in Volume
Example:	4 gal. per minute	2 min.	Increase of 8 gallons
1.	2 gal. per minute out of the tank	2 min.	
2.	1 gal. per minute OUT	5 min.	
3.	6 gal. per minute IN	3 min.	
4.	3 gal. per minute OUT	4 min.	
5.	10 gal. per minute IN	10 min.	
6.	2 gal. per minute IN	7 min.	
7.	12 gal. per minute OUT	1 min.	

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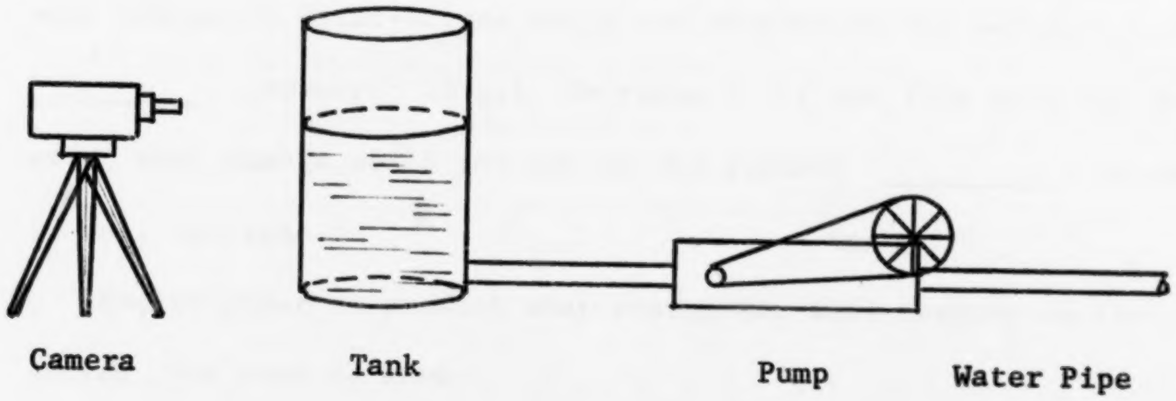
C. Imagine a camera which takes a movie of the tank while the pump is operating. Here is a picture of the situation we have imagined thus far.



Suppose the pump and camera are turned on together, and that 3 gallons of water flow into the tank each minute. After the pump and camera have run for 2 minutes, they are stopped. The film is then developed and projected on a screen. What change in the water-volume do you observe on the screen? It is easy to predict that the change observed on the screen will be a change of 6 gallons. But, will it be an increase or a decrease? __________.

In order to answer this question, you need to know whether the film is being run forward through the projector or backward through the projector.

Suppose the film was run backward through the projector. (Have you ever watched a comedy film in which a man seems to dive up out of the water and land on a diving board, or a race horse runs backward on a muddy track, picking up its footprints as it goes?) If the film were run backward, what change in water-volume would you see on the screen? __________. (Answer: You would see a decrease of 6 gallons on the screen.)

Now suppose the water was being pumped out of the tank at the rate of 5 gallons per minute, and both the pump and camera ran for 2 minutes. If the film were fun forward through the projector, what change in water-volume would you observe on the screen? __________ (Answer: 10 gal. decrease.) If the film were run backward, what change would you see on the screen? __________ (Answer: 10 gal. increase.)

So, in order to predict what change you will observe on the screen, you need to know:

1. The amount of water per minute being pumped into or out of the tank, and
2. The number of minutes the film is being run forward or backward.

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

D. __Exercises__: Complete the table on the next page. The first problem has been solved for you as an example, and is shown here.

	Pump	Movie	Observed Change in Volume
1.	Filling 5 gal. per minute.	Running backward 3 minute	Decrease of 15 gallons

In problem 1, you are told that the pump is filling (forcing water INTO) the tank at the rate of 5 gallons per minute. From the second column you learn that the movie has been run backward for 3 minutes. Since the pump is filling the tank, and since the film is run backward, the volume of water appears to be decreasing (just the opposite of what actually happened in the tank). So we see on the screen a decrease in volume of 15 gallons, which we write in the box provided.

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

	Pump	Movie	Observed Change in Volume
1.	Filling 5 gal. per minute	Running backward 3 minutes	Decrease of 15 gallons
2.	Filling 1 gal. per minute	Running forward 13 minutes	
3.	Emptying 4 gal. per minute	Running forward 2 minutes	
4.	Filling 10 gal. per minute	Running backward 5 minutes	
5.	Emptying 3 gal. per minute	Running backward 4 minutes	
6.	Emptying 11 gal. per minute	Running forward 3 minutes	
7.	Filling 2 gal. per minute	Running forward 6 minutes	
8.	Filling 5 gal. per minute	Running backward 2 minutes	
9.	Emptying 5 gal. per minute	Running forward 2 minutes	
10.	Filling 5 gal. per minute	Running forward 2 minutes	
11.	Emptying 5 gal. per minute	Running backward 2 minutes	

E. Look! New numerals! Complete the table.

	Pump	Movie	Observed Change in Volume
1.	Filling 5 gal. per minute	Running forward 4 minutes	Increase of 20 gallons
	+5	+4	+20
2.	Filling 3 gal. per minute	Running backward 4 minutes	Decrease of 12 gallons
	+3	-4	-12
3.	Emptying 8 gal. per minute	Running forward 2 minutes	Decrease of 16 gallons
	-8	+2	-16
4.	Emptying 4 gal. per minute	Running backward 10 minutes	Increase of 40 gallons

F. We know that real numbers are numbers which measure both distance (amount) and direction. In general, anything which involves an amount and one of two opposite directions can be measured by a real number. Our pump, tank, film situation involves amounts (the rate of water flow and number of minutes) and direction (filling or emptying the tank, running the film forward or backward) and therefore can be measured by real numbers. We can use real numbers to measure the rate at which the water is being pumped, deciding to use

positive numbers when water flows into the tank,

and

negative numbers when water flows out of the tank.

Thus, if 4 gallons of water are being pumped into the tank each minute, we say that the rate is $^{+}4$ gallons per minute. Explain what is meant by saying that the rate is $^{-}4$ gallons per minute.

____

____.

Also, we can use real numbers to measure how long the film is being projected, deciding to use

positive numbers when the film is run forward,

and

negative numbers when the film is run backward.

So, "The film is running for $^{+}3$ minutes," will mean "The film is being projected forward (normally) for 3 minutes." Explain what is meant by "The film is running for $^{-}3$ minutes." ________________

____

When we use a numeral such as

$$"{}^{-}4 \times {}^{+}3"$$

We will think of $^{-}4$ as measuring the rate at which water is being pumped. We will think of $^{+}3$ as telling us how long the film is being projected. And, finally, we will think of the product

$$^{-}4 \times {}^{+}3$$

as measuring the change in water-volume which we see on the screen. (Let's agree to use *positive* numbers to measure observed increases on the screen and *negative* numbers to measure observed *decreases*.) In this case, since water is being pumped *out* of the tank at 4 gallons per minute and the film is being run forward for 3 minutes, the change in water-volume observed on the screen is a *decrease* of 12 gallons. So, the change is $^{-}12$ gallons. Since we agreed to think of the product as a measure of the change, we can say that

$$^{-}4 \times {}^{+}3 = {}^{-}12.$$

Let's take another case:

$$^{+}9 \times {}^{-}8 = ?$$

This number, $^{+}9 \times {}^{-}8$, measures the change in water-volume which you would observe on a screen. The first number, $^{+}9$, measures the rate of pumping (is it filling or is it emptying?), and the second number, $^{-}8$, tells how long the film is being projected (is it being run forward or is it being run backward?). Does $^{+}9 \times {}^{-}8$ measure an observed *increase* in water-volume or an observed *decrease*? ________

Since $^{+}9$ would indicate that 9 gallons of water are being pumped *into* the tank each minute, and $^{-}8$ would indicate the film being run *backward* for 8 minutes, we would observe a *decrease* in water volume of 72 gallons. Therefore, we can say that

$$^{+}9 \times {}^{-}8 = {}^{-}72$$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

G. Exercises

The table below contains problems dealing with the pump-tank-movie interpretation. We have solved the first problem for you as a sample. Complete the table.

	Pump	Movie	Observed Change in Volume
1.	Filling--1 gal. per minute	Running backward 17 minutes	Decrease of 17 gallons
	+1	−17	−17
2.	Filling--11 gal. per minute	Running forward 2 minutes	
3.	Emptying--2 gal per minute	Running forward 8 minutes	
4.	Filling--3 gal. per minute	Running backward 2 minutes	
5.	Emptying--4 gal. per minute	Running backward 5 minutes	
6.	Emptying--6 gal. per minute	Running forward 3 minutes	
7.	Filling--6 gal. per minute	Running forward 10 minutes	

	Pump	Movie	Observed Change in Volume
8.	Filling--7 gal. per minute	Running backward 2 minutes	
9.	Emptying—100 gal. per minute	Running backward 9 minutes	
10.	Emptying--6 gal. per minute	Running forward 6 minutes	

(Note: In the rest of the problems you are given real numbers and you should fill in the corresponding blanks. Problem 11 has been worked for you as an example.)

11.	Filling – 4 gallons per minute	Running forward 3 minutes	increase of 12 gallons
	$^{+}4$	$^{+}3$	+12
12.			
	$^{-}7$	$^{+}2$	
13.			
	$^{-}11$	$^{-}5$	

	Pump	Movie	Observed Change in Volume
14.			
	$^{+}10$	$^{-}6$	
15.			
	$^{+}5$	$^{-}4$	
16.			
	$^{-}5$	$^{+}4$	
17.			
	$^{+}5$	$^{+}4$	
18.			
	$^{-}5$	$^{-}4$	

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

H. Exercises

The table below contains problems dealing with the pump-tank-movie interpretation. From each problem you can learn how to multiply a pair of real numbers. We have solved the first problem for you as a sample. Complete the table.

<table>
<tr><th></th><th>Pump</th><th>Movie</th><th>Observed Change in Volume</th></tr>
<tr><td>1.</td><td>Filling—4 gal. per minute</td><td>Running backward—2 minutes</td><td>Decrease of 8 gallons</td></tr>
<tr><td></td><td>$^{+}4$</td><td>$^{-}2$</td><td>$^{-}8$</td></tr>
<tr><td></td><td colspan="3">Corresponding multiplication statement: $^{+}4 \times {}^{-}2 = {}^{-}8$</td></tr>
<tr><td>2.</td><td>Emptying--4 gal. per minute</td><td>Running forward—2 minutes</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="3">Corresponding multiplication statement:</td></tr>
<tr><td>3.</td><td>Filling—4 gal. per minute</td><td>Running forward—2 minutes</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="3">Corresponding multiplication statement:</td></tr>
<tr><td>4.</td><td>Emptying—4 gal. per minute</td><td>Running backward—2 minutes</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="3">Corresponding multiplication statement:</td></tr>
</table>

	Pump	Movie	Observed Change in Volume
5.	Filling--8 gal. per minute	Running forward--3 minutes	
	Corresponding multiplication statement:		
6.	Emptying--3 gal. per minute	Running forward--8 minutes	
	Corresponding multiplication statement:		
7.	Emptying--8 gal. per minute	Running backward--3 minutes	
	Corresponding multiplication statement:		

(Note: In the rest of the problems you are given real numbers and you should fill in the corresponding blanks.)

8.			
	$^{-}5$	$^{-}6$	
	Corresponding multiplication statement: $^{-}5$ x $^{-}6$ =		

	Pump	Movie	Observed Change in Volume
9.			
	$^{+}7$	$^{-}3$	
	Corresponding multiplication statement: $^{+}7 \times {}^{-}3 =$		
10.			
	$^{-}8$	0	
	Corresponding multiplication statement: $^{-}8 \times 0 =$		
11.			
	$^{-}6$	$^{-}4$	
	Corresponding multiplication statement: $^{-}6 \times {}^{-}4 =$		

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

I. Simplify. Use the pump-tank-film interpretation.

1. $^{+}2 \times {}^{+}5$
2. $^{+}7 \times {}^{+}4$
3. $^{+}8 \times {}^{-}8$
4. $^{-}3 \times {}^{-}6$
5. $^{+}6 \times {}^{-}2$
6. $^{-}2 \times {}^{+}6$
7. $^{+}5 \times {}^{+}7$
8. $^{+}8 \times {}^{+}8$
9. $^{+}9 \times {}^{+}10$
10. $^{+}2 \times {}^{-}10$
11. $^{-}7 \times {}^{+}8$
12. $^{+}15 \times {}^{-}3$
13. $^{+}1 \times {}^{-}1$
14. $^{-}8 \times {}^{+}12$
15. $^{+}7 \times {}^{-}2$
16. $^{-}6 \times {}^{-}4$
17. $^{-}100 \times {}^{-}2$
18. $^{-}5 \times {}^{-}10$
19. $^{-}3 \times {}^{-}15$
20. $0 \times {}^{-}7$
21. $^{+}6 \times 0$
22. 0×0
23. $^{-}12 \times 0$
24. $^{-}4 \times {}^{-}1$
25. $^{-}7 \times {}^{+}1$
26. $^{-}3 \times {}^{-}8$
27. $^{+}6 \times {}^{-}5$
28. $^{-}3 \times {}^{-}2$
29. $^{-}8 \times {}^{+}2$
30. $^{+}7 \times {}^{-}8$
31. $^{+}7 \times {}^{-}6$
32. $^{+}75 \times {}^{+}5$
33. $^{-}86 \times {}^{-}10$
34. $(^{+}2 \times {}^{-}3) \times {}^{-}4$
35. $(^{-}2 \times {}^{+}7) \times {}^{-}3$
36. $(^{+}5 \times {}^{-}3) \times {}^{-}4$
37. $(^{+}6 \times {}^{-}2) \times {}^{-}3$
38. $^{+}4 \times (^{-}3 \times {}^{-}7)$
39. $^{-}6 \times (^{+}2 \times {}^{-}5)$
40. $(^{+}73 \times {}^{-}81) \times 0$
41. $(^{+}5 \times {}^{-}17) \times (^{-}3 \times 0)$
42. $^{-}981 \times 0$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

J. If you have not yet found a short cut, use the pump-tank-film interpretation on the following problems, then look carefully at the results.

1. ${}^{+}2 \times {}^{+}2$
2. ${}^{-}2 \times {}^{+}2$
3. ${}^{+}2 \times {}^{-}2$
4. ${}^{-}2 \times {}^{-}2$
5. ${}^{+}8 \times {}^{-}5$
6. ${}^{+}8 \times {}^{+}5$
7. ${}^{-}8 \times {}^{+}5$
8. ${}^{-}8 \times {}^{-}5$

9. ${}^{+}3 \times {}^{+}3$
10. ${}^{+}3 \times {}^{-}3$
11. ${}^{-}3 \times {}^{+}3$
12. ${}^{-}3 \times {}^{-}3$
13. ${}^{+}5 \times {}^{+}5$
14. ${}^{+}5 \times {}^{-}5$
15. ${}^{-}5 \times {}^{+}5$
16. ${}^{-}5 \times {}^{-}5$
17. ${}^{+}10 \times {}^{+}2$
18. ${}^{+}10 \times {}^{-}2$
19. ${}^{-}10 \times {}^{+}2$
20. ${}^{-}10 \times {}^{-}2$

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K. Simplify Be careful not to confuse addition signs with multiplication signs.

1. $(^{+}5 + {}^{-}3) \times {}^{-}7$
2. $(^{+}3 \times {}^{-}4) + {}^{-}6$
3. $(^{+}8 \times {}^{-}3) + {}^{-}5$
4. $(^{+}12 + {}^{-}11) \times {}^{-}3$
5. $(^{+}1 + {}^{+}1) + {}^{+}1$
6. $(^{+}1 \times {}^{+}1) \times {}^{+}1$
7. $(^{-}1 + {}^{-}1) + {}^{-}1$
8. $(^{-}1 \times {}^{-}1) \times {}^{-}1$
9. $(^{-}4 \times {}^{-}2) + (^{-}5 \times {}^{+}6)$
10. $(^{-}3 \times {}^{+}7) + (^{+}8 \times {}^{-}4)$
11. $(^{+}71 + {}^{-}11) \times (^{+}6 + {}^{-}4)$
12. $(^{-}8 + 0 \times (^{+}4 \times {}^{+}2)$

L. Fill in the blanks to make true sentences.

1. $^{+}5 \times$ _____ $= {}^{-}20$
2. $^{-}3 \times$ _____ $= {}^{+}18$
3. $^{+}7 \times$ _____ $= {}^{+}21$
4. $^{-}7 \times$ _____ $= {}^{-}21$
5. _____ $+ {}^{-}3 = {}^{+}9$
6. _____ $\times {}^{-}3 = {}^{+}9$
7. $^{+}8 +$ _____ $= {}^{-}8$
8. $^{+}8 \times$ _____ $= {}^{-}8$
9. $^{+}3 \times$ _____ $= {}^{+}1$
10. $^{+}3 +$ _____ $= 0$
11. $^{+}3 \times$ _____ $= 0$
12. $^{+}3 +$ _____ $= {}^{+}1$
13. $^{+}8 \times {}^{-}2 = {}^{-}(8 \times$ _____$)$
14. $^{-}5 \times$ _____ $= {}^{+}(5 \times 2)$

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III. Exploration Exercises--dpD

A. Simplify.

Example: $6 \times 5 + 7 \times 5$

Solution:
$$\begin{aligned} 6 \times 5 + 7 \times 5 &= (6 \times 5) + (7 \times 5) \\ &= 30 + 35 \\ &= 65 \end{aligned}$$

1. $3 \times 9 + 8 \times 9$
2. $8 \times 83 + 8 \times 17$
3. $5 \times 13 + 5 \times 17$
4. $13 \times 5 + 15 \times 5$
5. $67 \times 25 + 33 \times 25$
6. $1 \times 6 + 99 \times 6$
7. $5 \times 987 + 5 \times 13$
8. $88 \times 21 + 12 \times 21$
9. $7 \times 4 + 7 \times 7$
10. $4 \times 19 + 4 \times 6$
11. $29 \times 3 + 4 \times 3$
12. $17 \times 5 + 3 \times 5$
13. $\frac{1}{5} \times 15 + \frac{4}{5} \times 15$
14. $11 \times 2\frac{1}{4} + 11 \times 2\frac{3}{4}$

B. Simplify mentally.

1. $3 \times 12 + 2 \times 12$
2. $6 \times 14 + 4 \times 14$
3. $3 \times 5 + 3 \times 4$
4. $3 \times 7 + 5 \times 7$
5. $3 \times 51 + 17 \times 51$
6. $47 \times 2 + 47 \times 8$
7. $12 \times 11 + 12 \times 1$
8. $11 \times 73 + 9 \times 73$
9. $138 \times 2 + 138 \times 8$
10. $29 \times 3 + 21 \times 3$
11. $4 \times 97 + 6 \times 97$
12. $13 \times 19 + 7 \times 19$
13. $19 \times 13 + 19 \times 7$
14. $163 \times 5 + 163 \times 5$
15. $163 \times 6 + 163 \times 4$
16. $9 \times 163 + 1 \times 163$

C. In each sentence fill the frames with numerals which will make the sentence true.

Remember when a frame of one shape appears more than once in an expression it must be filled by a numeral for the same number each time. For example, in the sentence:

$$\square + \square = 14, \ \boxed{7} + \boxed{7} = 14$$

is allowed, but $\boxed{6} + \boxed{8} = 14$ is not allowed.

1. $6 \times \square + 4 \times \square = 10$
2. $6 \times \square + 4 \times \square = 30$
3. $6 \times \square + 4 \times \square = 60$
4. $6 \times \square + 4 \times \square = 90$
5. $6 \times \square + 4 \times \square = 110$
6. $6 \times \square + 4 \times \square = 250$
7. $6 \times \square + 4 \times \square = 20$
8. $6 \times \square + 4 \times \square = 1000$
9. $6 \times \square + 4 \times \square = 22$
10. $6 \times \square + 4 \times \square = 311$
11. $3 \times \square + 7 \times \square = 80$
12. $3 \times \square + 7 \times \square = 50$
13. $8 \times \square + 2 \times \square = 70$
14. $98 \times \square + 2 \times \square = 200$
15. $5 \times \square + 7 \times \square = 48$
16. $4 \times \square + 7 \times \square = 22$
17. $6 \times \square + 3 \times \square = 90$
18. $87 \times \square + 13 \times \square = 500$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

D. Simplify each expression listed below, then give a whole number (other than itself and 1) which divides it exactly. For example: 2, 3, 6, 9 each divides 18 exactly, but 7 does not divide 18 exactly.

Example 1: 21 + 5

Solution 1: This simplifies to 26. 13 is a number which divides 21 + 5 exactly. Another number which divides 21 + 5 exactly is 2. Either is a satisfactory answer.

Example 2: 3 x 6 + 7 x 6

Solution 2: This simplifies to 60. 6 is a number which divides 3 x 6 + 7 x 6 exactly. Other numbers which divide 3 x 6 + 7 x 6 exactly are 2, 3, 4, 5, 10, 12, 15, 20, and 30. Any one of these is a satisfactory answer.

1. 3 x 6
2. 21 + 3
3. 13 + 1
4. 5 x 7
5. 4 x 8 + 3
6. 3 x 5 + 4 x 5
7. 8 x 17 + 2 x 17
8. 12 x 4 + 12 x 16
9. 2 x 21 + 6 x 21
10. 91 x 7 + 9 x 7
11. 374 x 2 + 374 x 2
12. 6 x 1 + 6 x 199
13. 7 x 6 + 4 x 6
14. 8 x 6 + 12 x 6

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

E. For each number listed below, give two numbers (other than itself and 1) which divide it exactly.

1. 3 x 13 + 7 x 13
2. 6 x 7 + 7 x 7
3. 4 x 22 + 10 x 22
4. 6 x 3 + 6 x 7
5. 8 x 7 + 2 x 7
6. 7 x 5 + 4 x 5
7. 5 x 3 + 5 x 8
8. 3 x 88 + 3 x 12
9. 1 x 4 + 999 x 4
10. 14 x 8 + 6 x 8
11. 7 x 5 + 7 x 4
12. 602 x 7 + 398 x 7
13. 4 x 14 + 4 x 11
14. 52 x 9 + 48 x 9
15. 97 x 6 + 97 x 4
16. 19 x 13 + 19 x 87

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

F. Complete these examples as rapidly and accurately as you can.

2 x 4 =	5 x 5 =	6 x 8 =
2 x 8 =	5 x 10 =	6 x 10 =
2 x 12 =	5 x 15 =	6 x 18 =
7 x 2 =	3 x 9 =	4 x 3 =
7 x 10 =	3 x 10 =	4 x 20 =
7 x 12 =	3 x 19 =	4 x 23 =

4 x3	10 x3	14 x3	6 x8	10 x8	16 x8	2 x7	10 x7	12 x7
9 x6	40 x6	49 x6	5 x5	70 x5	75 x5	6 x7	20 x7	26 x7
5 x9	50 x9	55 x9	8 x4	20 x4	28 x4	5 x3	30 x3	35 x3
9 x2	60 x2	69 x2	10 x4	11 x4	21 x4	10 x8	9 x8	19 x8

4 x 9 =	7 x 5 =	9 x 3 =
4 x 40 =	7 x = 420	9 x = 450
4 x 49 =	7 x = 455	9 x = 477
7 x 3 =	8 x 8 =	4 x = 28
7 x 70 =	8 x 70 =	4 x = 280
7 x 73 =	8 x 78 =	4 x = 308

G. Complete these examples as rapidly and accurately as you can.

3 x6	50 x6	53 x6	4 x7	50 x7	54 x7	3 x9	40 x9	43 x9
9 x4	60 x4	69 x4	6 x6	40 x6	46 x6	5 x7	60 x7	65 x7
4 x5	40 x5	44 x5	9 x5	80 x5	89 x5	8 x9	70 x9	78 x9
6 x4	90 x4	96 x4	9 x8	70 x8	79 x8	7 x7	90 x7	97 x7
9 x9	90 x9	99 x9	5 x4	80 x4	85 x4	6 x8	90 x8	96 x8
8 x2	70 x2	78 x2	3 x8	80 x8	83 x8	5 x6	80 x6	85 x6
8 x8	50 x8	58 x8	6 x3	80 x3	86 x3	7 x9	50 x9	57 x9
4 x6	70 x6	74 x6	3 x5	90 x5	93 x5	8 x5	50 x5	58 x5
7 x3	90 x3	97 x3	6 x9	80 x9	86 x9	8 x3	60 x3	68 x3
4 x4	60 x4	64 x4	2 x6	90 x6	92 x6	9 x7	80 x7	89 x7

H. Complete the following examples.

3 x 2 = 6 x 3 = 4 x 5 =

3 x 10 = 6 x 20 = 4 x 20 =

3 x 300 = 6 x 100 = 4 x 300 =

3 x 312 = 6 x 123 = 4 x 325 =

5 x5	20 x5	300 x5	325 x5	7 x9	00 x9	400 x9	407 x9
7 x2	10 x2	500 x2	517 x2	0 x7	90 x7	800 x7	890 x7
8 x5	40 x5	700 x5	748 x5	9 x6	40 x6	700 x6	749 x6
6 x4	40 x4	800 x4	846 x4	1 x8	90 x8	200 x8	291 x8
8 x6	50 x6	100 x6	158 x6	4 x7	00 x7	600 x7	604 x7
9 x3	40 x3	800 x3	849 x3	7 x8	40 x8	600 x8	647 x8
6 x5	50 x5	200 x5	256 x5	0 x9	10 x9	200 x9	210 x9
8 x8	50 x8	300 x8	358 x8	9 x9	50 x9	300 x9	359 x9

Is it easier to work the examples in the boxes before you have worked the other examples or after you have worked them? ______________

____

I. On this page you need to help make up some of the examples in addition to working them. Note that there is a certain way to make the example.

9 x4	60 x4	69 x4	6 x3	70 x3	76 x3	5 x7	90 x7	95 x7
4 x5	40 x5	44 x5	7 x2	80 x2	87 x_	9 x3	10 x3	19 x_
6 x4	90 x4	96 x_	9 x9	90 x9	x9	8 x7	50 x7	x7
6 x5	70 x5	76 x5	4 x4	40 x4	44 x4	10 x8	7 x8	17 x_
7 x7	30 x7		20 x6	6 x6		9 x8	50 x8	
3 x7	20 x7		9 x5	100 x5		40 x10	18 x10	

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

J. Work the examples in the heavy outlined boxes. You do not need to do the others, but you may find them helpful.

2 x 14 = 28 5 x 16 = 3 x 21 =

10 x 14 = 140 10 x 16 = 20 x 21 =

$\boxed{12 \text{ x } 14 = \qquad 15 \text{ x } 16 = \qquad 23 \text{ x } 21 =}$

4 x 12 = 6 x 15 = 1 x 23 =

10 x 12 = 10 x 15 = 20 x 23 =

$\boxed{14 \text{ x } 12 = \qquad 16 \text{ x } 15 = \qquad 21 \text{ x } 23 =}$

$\begin{array}{r}42\\ \underline{\text{x}9}\end{array}$ $\begin{array}{r}42\\ \underline{\text{x}10}\end{array}$ $\boxed{\begin{array}{r}42\\ \underline{\text{x}19}\end{array}}$ $\begin{array}{r}48\\ \underline{\text{x}7}\end{array}$ $\begin{array}{r}48\\ \underline{\text{x}20}\end{array}$ $\boxed{\begin{array}{r}48\\ \underline{\text{x}27}\end{array}}$

2 x 19 = 4 x 25 = 2 x 35 =

40 x 19 = 30 x 25 = 60 x 35 =

$\boxed{42 \text{ x } 19 = \qquad 34 \text{ x } 25 = \qquad 62 \text{ x } 35 =}$

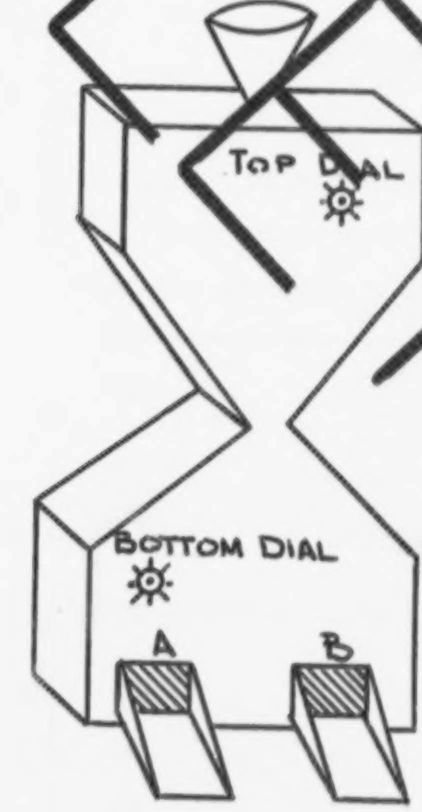
$\begin{array}{r}34\\ \underline{\text{x}5}\end{array}$ $\begin{array}{r}34\\ \underline{\text{x}20}\end{array}$ $\boxed{\begin{array}{r}34\\ \underline{\text{x}25}\end{array}}$ $\begin{array}{r}62\\ \underline{\text{x}5}\end{array}$ $\begin{array}{r}62\\ \underline{\text{x}30}\end{array}$ $\boxed{\begin{array}{r}62\\ \underline{\text{x}35}\end{array}}$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

K. John Dawson came to Mr. Hall's science class one Monday and announced, "I've just invented a new machine!"

"What does it do?" asked Mr. Hall.

"Well," answered John, "When you drop several objects into the funnel on top, the machine ties them into bundles and pushes the bundles out two little chutes in the front. Here, let me show you how it works. I'll set the top dial on '5' and the bottom dial on '3'."



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"Here Susan, hand me that bag of marbles on the shelf," requested John.

John dropped 40 marbles into the funnel on the top of the machine. The machine vibrated and groaned, and suddenly 3 bundles of marbles dropped out of chute A and 5 bundles dropped out of chute B. They unwrapped the bundles and found 5 marbles in each bundle.

Susan said, "Let's change the top dial to '10', then put all 40 marbles in again."

"Wait," exclaimed Mr. Hall, "Let's keep a record of what happens each time." He started a record on the blackboard that looked like this.

Dial Settings		Number of Marbles		Number of Bundles	
TOP	BOTTOM	Put In	In Each Bundle	Out At A	Out At B
5	3	40	5	3	5
10	3	40	10	3	[illegible]
2	3	40	2	3	17
4	3	40	4		7
8	3	40		3	2

As the machine worked, they changed the top dial each time, and continued to record the results. They left some blank for you to fill in.

John accidentally dropped the marbles on the floor and 4 of them rolled under the bookcase.

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"Shucks," said Harry, "Now we can't go on!"

"Sure we can," replied John, "We can drop in 36 marbles just as well. Let's set the top dial at '4' this time."

Dial Settings		Number of Marbles		Number of Bundles	
TOP	BOTTOM	Put In	In Each Bundle	Out At A	Out At B
4	3	36	4	3	6
12	3	36			0
6		36		3	

Suddenly the class was disturbed by a muffled voice from under the bookcase. "Hey, I found those 4 marbles and a whole bunch more," Stan exclaimed as he crawled out and wiped dust from his clothes. "We've got over 60 marbles now!"

The class now experimented dropping various numbers of marbles into the machine. They also experimented with various settings for the bottom dial. Complete the record wherever they forgot to fill it in.

Dial Settings		Number of Marbles		Number of Bundles	
TOP	BOTTOM	Put In	In Each Bundle	Out At A	Out At B
6	2	60	6	2	8
5	1	35	5	1	6
8	5	48		5	1
2	15	50	2		10

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Dial Settings		Number of Marbles		Number of Bundles	
TOP	BOTTOM	Put In	In Each Bundle	Out At A	Out At B
4	[illegible]	32			
5	2	41	5	2	6

On the next trial, the machine wouldn't work. Finally, John removed the cover and found 1 marble from the previous trial stuck in the machine.

"No wonder it didn't work! We didn't use the right number of marbles last time," commented Mary. "We put in 41 marbles and only 40 came out."

"Let's keep track of how many marbles come out each time," suggested Bill, "so we can see if any of them get stuck."

Mr. Hall added two extra columns and the class continued working with the machine. They left some of the spaces blank, hoping you would fill them in.

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Dial Settings		Number of Marbles		Number of Bundles		TOTALS OUT	
TOP	BOTTOM	Put In	In Each Bundle	Out At A	Out At B	Bundles	Marbles
2	20	58	2	20	9	29	58
4	6	44	4	6	5	11	44
4	3	20		3	2	5	20
6	2	30	6	2	3	5	30
7	1	49	7		6	7	
5	8	56			3		
	6	48		6	2	8	48
3	7	60					
20		51		1		2	

Put a check in front of each trial which left any marbles stuck in the machine.

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L. Fill in the frames to make true sentences. Simplify as needed.

Example: 2 x 6 + 8 x 6 = (2 + 8) x □

Solution: (2 x 6) + (8 x 6) = 10 x □

12 + 48 = 10 x □

60 = 10 x □

Therefore, if you fill the frame with a '6,' it would make the sentence true.

1. 7 x 8 + 3 x 8 = (7 + □) x 8
2. 5 x 6 + 3 x 6 = (5 + 3) x △
3. ⬡ x (7 + 8) = ⬡ x 7 + ⬡ x 8
4. 8 x 5 + 2 x 5 = (□ + 2) x 5
5. (6 + 4) x 5 = (△ + 2) x 5
6. 7 x (3 + 2) = 7 x 3 + 7 x □

(Remember our agreement about sentences with two or more frames of the same shape. Each frame must be filled with a numeral for the same number.)

7. 5 x (4 + 8) = □ x 4 + □ x 8
8. 6 x 9 + △ x 9 = (6 + △) x 9
9. 3 x 16 + 7 x 16 = ⬡ x 16
10. 5 x 19 + 5 x 19 = (5 + 5) x △
11. 11 x 4 + 11 x □ = 11 x 7
12. 53 x ⬡ + 53 x 17 = 53 x (13 +17)
13. △ x 9 + △ x 11 = 132 x (9 + 11)
14. 87 x (93 + □) = 87 x 93 + 87 x □
15. 561 x 432 + 413 x 432 = (△ + 413) x 432

M. Perhaps you have found "short cuts" for some problems which involve both multiplication and addition.

7 x 12 + 3 x 12 = ?

Do you see a short way of solving this problem? If you don't, you may see it after you have filled in the blanks in the following sentences.

4 x 11 + 6 x 11 = 44 + 66 = 110 = 10 x 11

8 x 23 + 2 x 23 = ____ + ____ = ____ = ____ x 23

5 x 9 + 6 x 9 = ____ + ____ = ____ = ____ x 9

From here on you need only to fill in the last two blanks for each problem.

Example: 2 x 17 + 8x17 = ____ + ____ = 170 = 10 x 17

You may fill in the other two blanks if you would like to.

17 x 15 + 3 x 15 = ____ + ____ = ____ = ____ x 15

9 x 4 + 21 x 4 = ____ + ____ = ____ = ____ x 4

91 x 8 + 9 x 8 = ____ + ____ = ____ = ____ x 8

$\frac{2}{3}$ x 12 + $\frac{1}{3}$ x 12 = ____ + ____ = ____ = ____ x 12

Can you do these problems by a short cut?

7 x 6 + 4 x 6 = ?

8 x 437 + 2 x 437 = ?

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

(This page should be used only after teacher presentation of the story of "Ben Harris," extracted from Wirtz, Robert W., Sawyer, W. W., and Botel, Morton, Math Workshop for Children, Level D. Encyclopedia Britannica Press, Inc., 1962, pp. 20-22.)

N. Ben's Rule

(Remember that when you multiply Ben's way you must imagine that you don't know any multiplication fact in which either factor is greater than 5.)

DO-IT-TO-BOTH

1. 2 x 9 = 2 x (____ + ____) = ____ + ____ = ____
2. 10 x 4 = (____ + ____) x 4____ = ____ + ____ = ____
3. 3 x 7 = ____ x (____ + ____) = ____ + ____ = ____
4. 8 x 3 = (____ + ____) x ____ = ____ + ____ = ____
5. 5 x 6 = ____ x (____ + ____) = ____ + ____ = ____
6. 2 x 10 = ____ x (____ + ____) = ____ + ____ = ____
7. 6 x 3 = (____ + ____) x ____ = ____ + ____ = ____
8. 2 x 7 = ____ x (____ + ____) = ____ + ____ = ____
9. 9 x 4 = (____ + ____) x ____ = ____ + ____ = ____
10. 3 x 10 = ____ x (____ + ____) = ____ + ____ = ____

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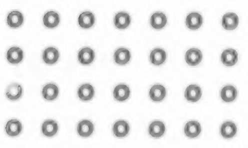
O. Why does Ben's "Do-It-To-Both" method seem to work? Here is a visual explanation.

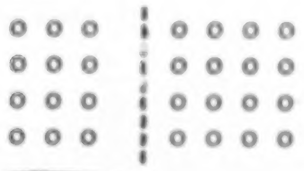
In counting dots:

Our Way



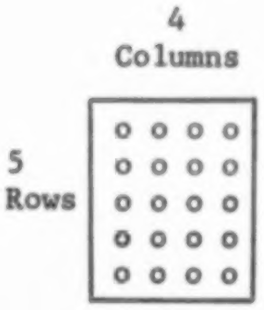
4 x 7 = 28

Ben's "Do-It-To-Both" Way



4 x 7 = 4 x (3 + 4) = 12 + 16 = 28

Our Way

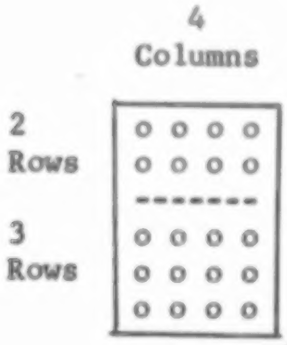


4 x 5 = 20

or

5 x 4 = 20

Ben's "Do-It-To-Both" Way



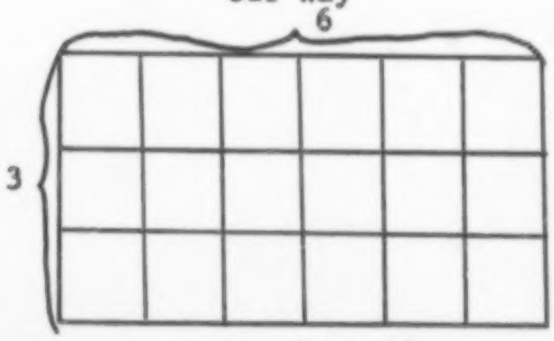
4 x 5 = 4 x (2 + 3) = 8 + 12 = 20

or

5 x 4 = (2 + 3) x 4 = 8 + 12 = 20

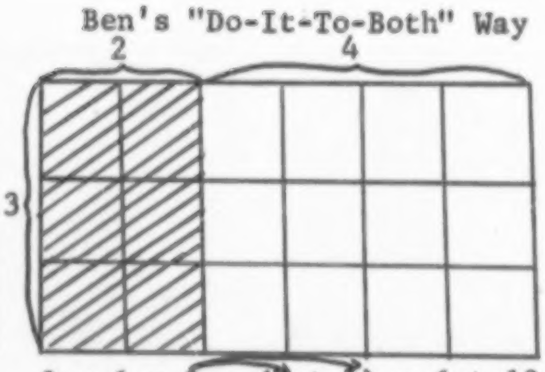
In counting spaces:

Our Way



3 x 16 = 18

Ben's "Do-It-To-Both" Way



3 x 6 = 3 x (2 + 4) = 6 + 12 = 18

P. Practice in "Do-It-To-Both"

1. $4 \times (2 + 5) = (4 \times 2) + (4 \times \_) = \_\_\_\_ + 20 = 28$
2. $(5 + 3) \times 3 = (\_ \times 3) + (3 \times 3) = 15 + \_\_\_\_ = 24$
3. $5 \times (5 + 2) = \_\_\_\_ + \_\_\_\_ = \_\_\_\_$
4. $3 \times (2 + 4) = \_\_\_\_ + \_\_\_\_ = \_\_\_\_$
5. $(4 + 5) \times 2 = \_\_\_\_ + \_\_\_\_ = \_\_\_\_$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

Q. Below are seven problems. The first three have been solved for you. Examine them carefully, then fill the blanks correctly in the last four problems.

1. 8 x 3 + 8 x 4 = 8 x 7 = 56
2. 7 x 6 + 7 x 4 = 7 x 10 = 70
3. 9 x 8 + 9 x 2 = 9 x 10 = 90
4. 4 x 9 + 4 x 1 = 4 x ____ = ____
5. 5 x 13 + 5 x 7 = ____ x ____ = ____
6. 6 x 49 + 7 x 51 = ____ x ____ = ____
7. 18 x 3 + 7 x 3 = 25 x ____ = ____

R. Simplify.

1. 1 x 8 + 9 x 8 =
2. 2 x 8 + 8 x 8 =
3. 3 x 8 + 7 x 8 =
4. 4 x 8 + 6 x 8 =
5. 5 x 8 + 5 x 8 =
6. 2 x 8 + 6 x 8 =

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S. Mark each of the following sentences true or false. Use the space on the right for any computations.

1. 18 x 4 + 7 x 4 = 100
2. 6 x 7 + 3 x 6 = 120
3. 5 x 2 + 5 x 8 = 100
4. 82 x 6 + 17 x 6 = 600
5. 5 x 49 + 5 x 51 = 500
6. 9 x 3 + 4 x 7 = 130
7. $\frac{1}{5} \times 91 + \frac{4}{5} \times 91 = 91$
8. 5 x 16 + 5 x 4 = 100
9. $99 \times \frac{1}{3} + 1 \times \frac{2}{3} = 100$
10. 3 x 7 + 7 x 3 = 100
11. 91 x 7 + 8 x 7 = 700

IV. Exploration Exercises--enD

A. Product expressions. The sentence $5 \times 6 = 30$ shows that 30 is the product of 5 and 6. Because it names a number as a product, an expression like 5×6 is called a product expression. Other product expressions for 30 are 10×3, 2×15, 30×1, and $2 \times 3 \times 5$.

B. Exercises. Write one or more product expression for each of the following numerals.

Example: $15 = 3 \times 5$

1. 18
2. 24
3. 20
4. 32
5. 25
6. 12

Write a simpler numeral for the number named by each of the following product expressions.

Example: $4 \times 7 = 28$

7. 3×9
8. 4×10
9. 3×15
10. 4×16
11. 6×30
12. 5×31
13. $2 \times 4 \times 4$
14. $4 \times 2 \times 2$
15. $5 \times 5 \times 5$
16. $2 \times 2 \times 2 \times 2 \times 3$
17. $2 \times 5 \times 5$
18. $4 \times 3 \times 2 \times 2$

C. Repeated factors. The sentence $5 \times 6 = 30$ shows that 5 and 6 are factors of 30. Other whole number factors of 30 are 10 and 3, 2 and 15, 30 and 1, and 2, 3 and 5.

D. Exercises.

1. How many times is the factor '2' used in the product expression in problem 18 in section B: ______. In Problem 16: ______.

2. What factor is repeated more than once in problem 17? ______

3. What number is shown as a repeated factor in problem 13 above?

    ______

4. Write one or more product expressions for each of the following numerals. Show at least one repeated factor in each product expression.

Example 1: 16 = 4 x 4

or

16 = 2 x 2 x 2 x 2

Example 2: 24 = 2 x 2 x 6

The number of blanks will help you with some of them, although you may find several which could be written other ways as well.

a. 12 = ____ x ____ x ____

b. 18 = ____ x ____ x ____

c. 20 = ____ x ____ x ____

d. 28 = ____ x ____ x ____

e. 27 = ____ x ____ x ____

f. 50 = ____ x ____ x ____

g. 49 = ____ x ____

h. 25 = ____ x ____

i. 32 = ____ x ____ x ____

j. 36 = ____ x ____

k. 44 = ____ x ____ x ____

l. 1000 = ____ x ____ x ____, or

m. 72 =

n. 90 = ____ x ____ x ____

o. 60 = ____ x ____ x ____

p. 125 = ____ x ____ x ____

q. 100 =

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E. Do you have an idea how the symbol 4^3 might be used or what it means?

4^3 is a symbol which names the same number that we usually express with the numeral 64. Therefore $4^3 = 64$.

$2^2 = 4$	$3^2 = 9$	$4^2 = 16$
$2^3 = 8$	$3^3 = 27$	$5^2 = 25$
$2^4 =$	$3^4 = 81$	$7^2 = 49$

F. **Exercises.** Write a numeral in each frame below to make the sentence true. (Remember: In any one sentence when you put a numeral in one frame, it must also be placed in each frame of the same shape in that sentence.)

Example: $6^2 = 36 =$ ⑥ x ⑥

1. $2^2 = 4 =$ △ x △
2. $2^3 = 8 =$ ○ x ○ x ○
3. $3^2 = 9 =$ □ x □
4. $2^4 = 16 =$ △ x △ x △ x △
5. $4^2 = 16 =$ □ x □
6. $3^3 = 27 =$ □ x □ x □
7. $3^4 = 81 =$ ○ x ○ x ○ x ○
8. $4^2 = 16 =$ △ x △
9. $5^2 = 25 =$ □ x □
10. $7^2 = 49 =$ ○ x ○

Fill the blanks with the numerals which will make the sentences true.

11. 343 = 7 x 7 x 7 = $\underline{\quad\quad}^{3}$
12. 32 = 2 x 2 x 2 x 2 x 2 = $\underline{\ 2\ }^{-}$
13. 100 = 10 x 10 = $\underline{\quad\quad}^{-}$
14. 144 = 12 x 12 = $\underline{\quad\quad}^{-}$
15. 1000 = 10 x 10 x 10 = $\underline{\quad\quad}^{-}$

G. Sort the following expressions into lists so that each expression in any list names the same number as every other expression in that list.

25 8^2 2^6 4 x 4 4^2

2^4 5^2 4 x 4 x 4 5 x 5

Names for 25	Names for 64	Names for 16

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

H. You have noticed that the symbol 4^3 is made up of two numerals. The large numeral at the left is called the base. What does it tell you? ____________________

The small numeral at the upper right is called the exponent. What does it tell you? ____________________

These new symbols (4^3, 7^2, 2^4, etc.) are called exponent forms. 4^3 is read "four to the third power." 7^2 is read "seven to the second power." 2^4 is read "two to the fourth power."

These exponent forms give us a shorter way to write product expressions which contain repeated factors.

In the example: $16 = 2 \times 2 \times 2 \times 2 = 2^4$, 16 is a decimal numeral. $2 \times 2 \times 2 \times 2$ is a product expression naming the same number, while 2^4 is an exponent form of the same number.

Some numbers have more than one exponent form. For example:

16 may be named by $2 \times 2 \times 2 \times 2 = 2^4$ or by

$4 \times 4 = 4^2$

64 may be named by $8 \times 8 = 8^2$ or by

$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$ or

$\_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ = \_\_\_\_^3$

81 may be named by $3 \times 3 \times 3 \times 3 = 3^4$ or by

$\_\_\_\_ \times \_\_\_\_ = \_\_\_\_^2$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

I. **Exercises.**

1. Write a product expression and decimal numeral for each of the following:

Example: $5^3 = 5 \times 5 \times 5 = 125$

a. 5^2 b. 2^5

c. 20^2 d. 3^4

e. 6^3 f. 10^3

2. Write an exponent form for each of the following product expressions:

Example: $14 \times 14 = 14^2$

a. $9 \times 9 \times 9 =$ ____.

b. $6 \times 6 \times 6 \times 6 =$ ____.

c. $19 \times 19 =$ ____.

d. $23 \times 23 \times 23 \times 23 \times 23 =$ ____.

e. $4 \times 4 \times 4 \times 4 \times 4 \times 4 =$ ____.

f. $11 \times 11 \times 11 \times 11 =$ ____.

3. Write, for each of the following, a name in exponent form. You may need to write a product expression before you can find the exponent form.

Example: $8 = 2 \times 2 \times 2 = 2^3$

a. 32 b. 81

c. 128 d. 144

e. 27 f. 243

4. Fill the blanks in the following sentences.

a. The numeral 5^2 has exponent ____ and base ____.

b. The numeral ____ has exponent 4 and base 3.

c. The numeral 4^3 is the exponent form of the product expression ____ X ____ X ____.

V. Exploration Exercises--meD

A. In this section you should multiply each of the following product expressions (consisting this time of two numbers expressed in exponent form), then express the answer in exponent form also. You will probably find it helpful to change the exponent forms to decimal numerals, multiply in the usual way and then change the answer back to exponent form.

Example: $2^1 \times 4^2 = ?$

Solution: $2^1 \times 4^2 = 2 \times (4 \times 4)$

$= 2 \times 16$

$= 32$

$32 = 2 \times 2 \times 2 \times 2 \times 2$

$2 \times 2 \times 2 \times 2 \times 2 = 2^5$

Therefore, $2^1 \times 4^2 = 2^5$

After you have worked several problems, you may find that you do not need to take all of these steps with every problem.

1. $2^2 \times 2^2 = 2^{-}$
2. $2^3 \times 2^2 = 2^{-}$
3. $3^1 \times 3^2 = 3^{-}$
4. $6^1 \times 6^1 = 6^{-}$
5. $2^1 \times 2^3 =$
6. $3^1 \times 3^3 =$
7. $3^2 \times 3^2 =$
8. $5^2 \times 5^1 =$
9. $2^4 \times 2^2 =$
10. $2^5 \times 2^1 =$
11. $9^1 \times 9^1 =$
12. $2^3 \times 2^3 =$
13. $4^2 \times 4^1 =$
14. $2^2 \times 2^1 =$
15. $2^4 \times 2^1 =$
16. $3^1 \times 3^1 =$
17. $2^2 \times 2^5 =$
18. $4^2 \times 4^2 =$
19. $4^1 \times 4^3 =$
20. $5^3 \times 5^1 =$

21. $3^2 \times 3^4 =$

22. $2^3 \times 2^4 =$

23. $3^3 \times 3^3 =$

24. $3^2 \times 3^3 =$

25. $2^6 \times 2^1 =$

26. $5^2 \times 5^2 =$

27. $27^2 \times 27^3 =$

28. $243^2 \times 243^2 =$

29. $1728^2 \times 1728^4 =$

30. $\triangle^2 \times \triangle^2$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

B. Fill the blanks so that the following sentences are true.

1. $3^1 \times 3^4 = 3^{\_}$ 2. $7^1 \times 7^1 = \_\_\_^2$
3. $6^2 \times 6^1 = \_\_\_^{\_}$ 4. $2^4 \times \_\_\_^2 = 2^6$
5. $3^3 \times 3^{\_} = 3^6$ 6. $5^{\_} \times 5^3 = 5^4$
7. $\_\_\_^2 \times 2^3 = 2^5$ 8. $18^2 \times 18^3 = \_\_\_^5$
9. $21^3 \times 21^6 = \_\_\_^{\_}$ 10. $\square^3 \times \square^4 = \_\_\_$

C. Sort the following symbols into pairs which name the same number.

Example: $3^2 \times 3^3$ is a name for the number 243.
$3^4 \times 3^1$ is also a name for the number 243.

Therefore, we have drawn a line between these two symbols, thus making a pair of symbols which name the same number, 243.

1. $3^2 \times 3^3$ 2. $2^5 \times 2^3$
3. $3^2 \times 3^{10}$ 4. $3^2 \times 2^4$
5. $2^2 \times 6^6$ 6. $3^4 \times 3^1$
7. $2^4 \times 2^4$ 8. $2^2 \times 2^4$
9. $5^2 \times 5^2$ 10. 5^4
11. $3^6 \times 3^6$ 12. $2^5 \times 2^1$

How many products did you need to compute to make your list of pairs? ______ If you had to multiply out more than two of the examples, you have not yet recognized how to make use of an important shortcut in multiplying numbers expressed in exponent form. Perhaps the following exercise will help you discover this shortcut.

D. Read through each example. Fill any blank so that the sentence is true.

1. $2^4 \times 2^2 = (2 \times 2 \times 2 \times 2) \times (2 \times 2) = 2^6$
$= 16 \times 4 = 64 = 2^6$

2. $3^2 \times 3^3 = (3 \times 3) \times (3 \times 3 \times 3) = \qquad = 3^5$

$9 \times 27 = 243 = 3^5$

3. $2^3 \times 2^3 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) = \qquad 2^6$

$= \_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_ = \underline{64} = \_\_\_$

4. $4^2 \times 4^3 = (4 \times 4) \times (\_\_\_ \times 4 \times \_\_\_) = \_\_\_^5$

5. $10^4 \times 10^2 = (10 \times 10 \times 10 \times 10) \times (10 \times 10) = \_\_\_$

You may have noticed that in each product expression ($2^4 \times 2^2$ for example) each exponent form has the same base. You may also have noticed that there is a relationship between the two exponents in the product expression and the exponent in the answer. (Example: In the product expression $2^4 \times 2^2$, the two exponents are 4 and 2. In the answer, 2^6, the exponent is 6. What relationship is there between 4 and 2 in the product expression and 6 in the answer. Hint: $4 + \boxed{} = 6$).

If you think that you now know the shortcut, check the three examples below to see if your shortcut would work on each example.

$8^3 \times 8^2 = 8^5 \qquad 4^6 \times 4^3 = 4^9 \qquad 19^3 \times 19^5 = 19^8$

Does your shortcut work? If not, ask your teacher to help you. If your shortcut does work, ask your teacher to give you exercise set E.

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

E. Complete each of the following to make true sentences. Use only exponent forms.

1. $2^1 \times 2^4 = \underline{\qquad}$
2. $3^2 \times \underline{\qquad} = 3^4$
3. $17^2 \times 17^4 = \underline{\qquad}$
4. $\underline{\qquad} \times 5^3 = 5^5$
5. $61^3 \times \underline{\qquad} = 61^{10}$
6. $20^5 \times 20^3 = \underline{\qquad}$
7. $111^3 \times 111^4 =$
8. $\underline{\qquad} \times 6^2 = 6^3$
9. $\square^5 \times \square^1 = \underline{\qquad}$
10. $\triangle^2 \times \triangle^5 = \underline{\qquad}$

F. Mark each of the following mathematical sentences true or false. You may make any necessary computations in the space to the right.

1. $10^2 \times 10^2 = 10^4$
2. $4^4 \times 4^3 = 4^7$
3. $3^3 = 3 \times 3$
4. $3^3 = 3 \times 3 \times 3$
5. $2^3 \times 2^4 = 4^7$
6. $2^3 \times 2^4 = 2^{12}$
7. $11^4 \times 11^2 = 11^6$
8. $2^3 + 2^4 = 2^7$
9. $2^4 - 2^3 = 2^1$
10. $5^3 = 3 \times 5$
11. $2^3 \div 2^2 = 2^1$
12. $2^4 \div 2^2 = 2^2$

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VI. Exploration Exercises--deD

A. Simplify the following expressions. Use only exponent forms as your final answer.

Example: $4^2 \div 2^2 = ?$

Solution:
$$\begin{aligned} 4^2 \div 2^2 &= (4 \times 4) \div (2 \times 2) \\ &= 16 \div 4 \\ &= 4 \\ &= 2^2 \end{aligned}$$

Therefore, $4^2 \div 2^2 = 2^2$

Continue to change the exponent forms to decimal numerals, divide, and change the answers back to exponent form as long as you find it helpful to do so.

1. $6^2 \div 3^2$
2. $2^4 \div 2^2$
3. $2^3 \div 2^1$
4. $3^3 \div 3^1$
5. $5^2 \div 5^1$
6. $2^5 \div 2^1$
7. $4^3 \div 4^1$
8. $2^5 \div 2^2$
9. $2^6 \div 2^3$
10. $3^3 \div 3^2$
11. $3^4 \div 3^2$
12. $2^6 \div 2^2$
13. $2^7 \div 2^2$
14. $4^4 \div 4^2$
15. $5^4 \div 5^2$
16. $3^6 \div 3^3$
17. $2^{10} \div 2^5$
18. $6^4 \div 6^1$
19. $10^4 \div 10^1$
20. $23^5 \div 23^2$
21. $104^4 \div 104^2$
22. $\bigcirc^6 \div \bigcirc^4$

B. Fill the blanks so that the following sentences are true.

1. $3^4 \div 3^1 = 3-$
2. $7^4 \div 7^2 = \_\_\_^2$
3. $6^5 \div 6^2 = \_\_\_^-$
4. $2^6 \div 2^- = 2^4$
5. $3^6 \div \_\_\_^3 = 3^3$
6. $5^- \div 5^1 = 5^3$
7. $\_\_\_^5 \div 2^3 = 2^2$
8. $43^5 \div 43^3 = \_\_\_^2$
9. $13^6 \div 13^2 = \_\_\_$
10. $⬡^9 \div ⬡^1 = \_\_\_$

By now you have probably recognized the use of an important shortcut in dividing numbers expressed in exponent form. Check the three examples below to see if your shortcut would work on each of them.

$8^6 \div 8^3 = 8^3 \qquad 3^{10} \div 3^4 = 3^6 \qquad 11^9 \div 11^2 = 11^7$

If your shortcut works on all three examples,ask your teacher to give you exercise set D. If your shortcut does not seem to work on all three examples or if you are at all uncertain about it, section C may be of help.

C. With product expressions using exponent forms containing the same base ($2^4 \div 2^2$ for example) there is a relationship between the exponents in the expression and the exponent in the answer. The example given above, $2^4 \div 2^2$ simplifies, when divided, to 2^2. In other words $2^4 \div 2^2 = 2^2$. What is the relationship between the exponents:

$$2^4 \div 2^2 = 2^2$$

Is it the same relationship which exists in the following examples?

$$7^6 \div 7^2 = 7^4$$

$$4^9 \div 4^4 = 4^5$$

$$8^3 \div 8^1 = 8^2$$

If you think that you now know the shortcut, check the sample below to see if it would work.

$$10^6 \div 10^1 = 10^5$$

If your shortcut works, ask for section D. If it doesn't work, ask your teacher to help you.

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D. Complete each of the following to make true sentences. Use only exponent forms.

1. $2^5 \div 2^2 =$ ____

2. $12^4 \div 12^2 =$ ____

3. $3^7 \div$ ____ $= 3^3$

4. ____ $\div 19^2 = 19^7$

5. $43^6 \div 43^1 =$

6. $\triangle^8 \div \triangle^7 =$ ____

E. Mark each of the following mathematical sentences true or false. You may make any necessary computations in the space to the right.

1. $9^4 \div 9^2 = 9^2$
2. $3^4 \div 3^2 = 0^2$
3. $4^6 \div 4^3 = 4^2$
4. $5^4 \div 5^2 = 5^2$
5. $6^3 \div 2^2 = 3^1$
6. $7^{11} \div 7^1 = 7^1$
7. $8^8 \div 8^5 = 8^3$
8. $10^{20} \div 10^5 = 10^{15}$

APPENDIX II

Materials Used With Treatment E—AN Expository Method

PAGE AND ABBREVIATION KEY FOR EACH UNIT IN TREATMENT E MATERIALS

raE - Real number addition, expository treatment A2.02

rmE - Real number multiplication, expository treatment . . A2.16

dpE - Distributive principle of multiplication over addition, expository treatment A2.38

enE - Exponential notation, expository treatment A2.59

meE - Multiplication of numbers expressed in exponential form, expository treatment A2.65

deE - Division of numbers expressed in exponential form, expository treatment A2.70

dpma - Distributive principle of multiplication over addition A2.38

Materials Used With Treatment E—An Expository Method

I. Real Number Addition--raE

A. The numbers with which you have been working in school since the first grade comprise a very useful set of numbers. Let us call them numbers of arithmetic. This set of numbers allows you to do many mathematical problems, such as

$10 - 8 = 2$ $\qquad$ $18 \div 3 = 6$ $\qquad$ $4 - 0 = 4$

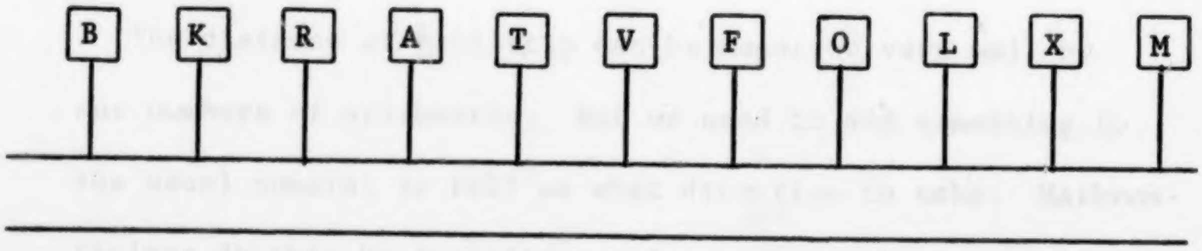
There are certain problems, however, which have no meaning when we use only numbers of arithmetic. For example,

$0 - 4 = ?$ $\qquad$ and $\qquad$ $8 - 10 = ?$

have no answers in the numbers of arithmetic. We shall introduce a new set of numbers which will give meaning to the problems 0 - 4 and 8 - 10. We shall call this set the real numbers.

B. We can best introduce the real numbers by giving you a physical model which shows the necessity for such numbers.

Imagine a road which has markers placed one mile apart at the side of the road. The markers are labeled with letters. The distance from R to A is one mile, for example. The distance from K to T is three miles.



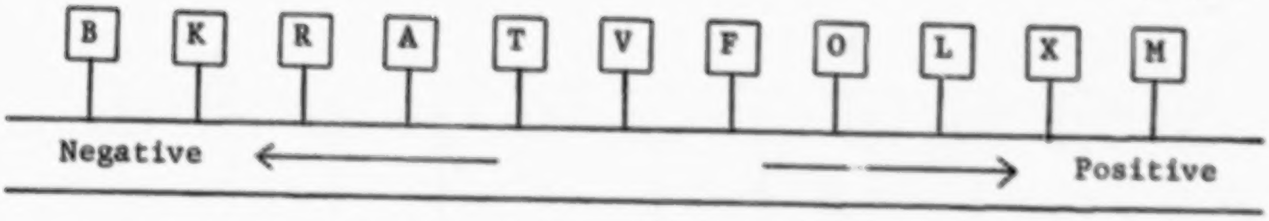
Below are two questions. Note that you cannot give a definite single answer to either of them. You must know more than the distance involved in each trip; you must also know something about the direction.

1. Suppose you are at the point F on this road and decide to take a trip that will take you three miles from F. Where will you be at the end of your trip? _______ (You cannot tell whether A or X is the correct answer.)
2. Suppose you and a friend are at the point T on this road. Each of you decides to take a trip that will take you two miles from T. How many miles apart will you be at the end of your trips? _______ Where on the road will you end your trips? Friend _______, You _______. (You cannot tell whether you will be four miles apart with one of you at R and the other at F, zero miles apart with both of you at R, or zero miles apart with both of you at F.)

The numbers of arithmetic can be used in measuring distances for trips along this road. But these numbers do not tell you in which direction the trips were made.

To be able to know what direction to take on this road as well as how far to travel, we need numbers which take into account both distance and direction. The set referred to earlier as real numbers takes both distance and direction into account.

The distance of each trip can be measured very well by our numbers of arithmetic. But we need to add something to the usual numeral to tell us what direction to take. Mathematicians do this by including a plus sign (+) or a minus sign (-). To illustrate this, let us draw the road again.



Note that we have indicated on the road which direction is positive and which direction is negative. Now, suppose you are at the point F and decide to take a trip of 3 miles in the positive direction. Where would you end up ? ______ (You would end up at X.) Such a trip could be indicated by the numeral $^{+}3$, in which the " + " tells you to move in the positive direction and the " 3 " tells you how far to go.

Suppose two people start at T and make two-mile trips; your trip being measured by $^{+}2$ and your friend's by $^{-}2$. Will you be together or four miles apart at the end of these trips? ______ (Since $^{+}2$ measures a trip of 2 miles in the positive direction this would take you to F and since $^{-}2$ measures a trip of 2 miles in the negative direction, this trip would take your friends to R. Therefore, at the end of these trips, you and your friend would be four miles apart.)

In working with numbers like $^{+}4$ and $^{-}6$, you will want to talk about them as well as write them. So you need to learn the correct pronunciation of their numerals. Mathematicians pronounce " $^{+}4$ " as "positive four" and " $^{-}6$ " as "negative six." Therefore, a trip from A to L is measured by a trip of positive five miles and could be called a "positive five trip." It would be indicated by the numeral $^{+}5$.

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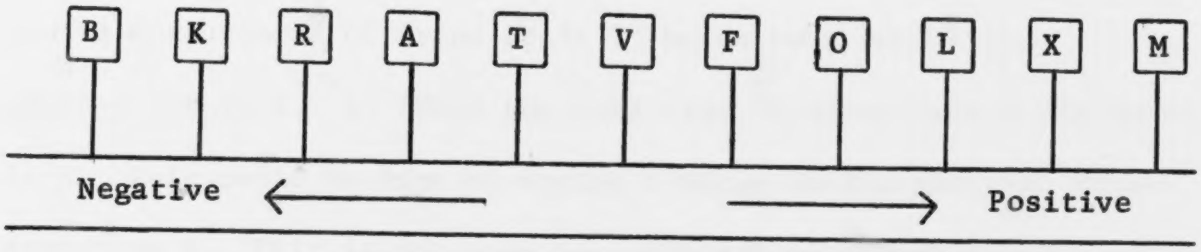
C.



1. Give the real numbers which measure the trips listed.

   Example: K to T

   Solution: $^{+}3$

   a. V to A　　b. A to V　　c. A to X

   d. A to F　　e. L to R　　f. B to R

2. List three trips which are measured by each of the given real numbers.

   Example: $^{+}2$

   Solution: K-A, T-F, O-X

   a. $^{-}3$　　b. $^{-}1$　　c. $^{+}8$

   d $^{-}5$　　e. $^{+}4$　　f. $^{+}6$

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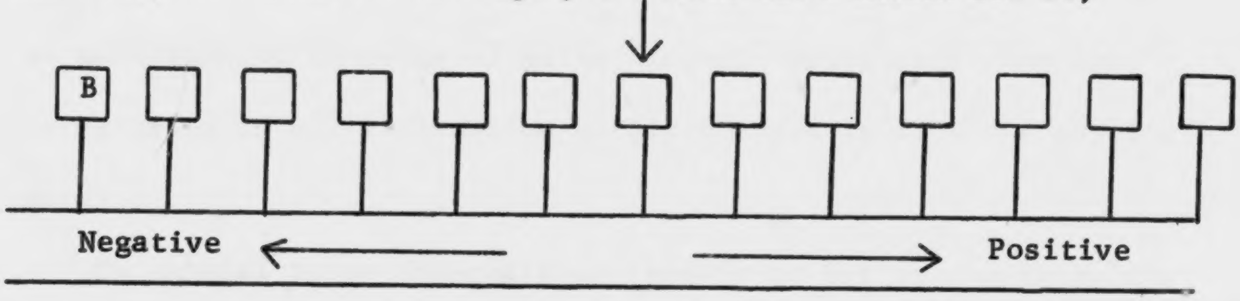
D. Imagine taking trips along a positive-negative road. Use the following measurement of trips to fill the markers correctly.

Example: B to S, $^{+}6$ (Find the road sign $^{+}6$ miles from B and label it S. This would be done by moving 6 miles in the positive direction from B. This is the sign you should have chosen for S.)



1. A trip from S to Y is measured by $^{-}4$

2. A trip from N to Y is measured by $^{-}2$

3. N to Q, $^{-}3$ 4. K to R, $^{-}2$

5. T to C, $^{-}2$ 6. Y to P, $^{+}5$

7. J to H, $^{-}2$ 8. Q to S, $^{+}5$

9. S to T, $^{+}5$ 10. R to C, $^{+}6$

11. J to T, $^{+}1$ 12. A to R, $^{-}9$

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E. Addition of Real Numbers.

To add $^{+}3$ and $^{+}5$, we think: 3 positive miles and 5 more positive miles give 8 positive miles, or $^{+}8$. Similarly, to add $^{-}3$ and $^{-}5$, we think: 3 negative miles and 5 more negative miles give 8 negative miles, or $^{-}8$. (Of course, this would hold true whether we were talking in terms of miles, inches, units, etc. We are using miles in our explanation merely because it fits one physical model.)

In general these observations lead us to the following rules:

Rules for Addition of Real Numbers

1. The sum of two real numbers with the same sign is a real number with this same sign and is equal to the sum of the numerical values of the two real numbers.

Since negative numbers are opposite to positive numbers, they have the effect of undoing one another That is to say, a number such as $^{-}7$ added to $^{+}7$ gives zero as an answer.

This leads us to add a second rule for addition of real numbers:

2. The sum of two real numbers of different signs but equal numerical value is zero.

In combining a positive number and a negative number of different numerical value, the smaller of the two can only partially undo the larger. Thus, $^{-}7$ added to $^{+}10$ leaves an excess of $^{+}3$; and $^{+}7$ added to $^{-}10$ leaves an excess of $^{-}3$.

This leads us to add the following rule:

3. The sum of two real numbers of different signs and unequal numerical values is a real number equal to the difference between the numerical values of the two numbers being added and has the sign of the real number of larger numerical value.

There is another rule which, though obvious, should be stated:

4. When zero is added to a real number, we get that real number as an answer; when zero is added to itself, the answer is zero.

To summarize what we have just learned, let us restate the above rules in a different way.

Suppose that "a" represents a first number of arithmetic and "b" represents a second number of arithmetic.

5. If a and b are numbers of arithmetic, then:

a. $(^{+}a) + (^{+}b) = ^{+}(a + b)$,

b. $(^{-}a) + (^{-}b) = ^{-}(a + b)$,

c. If $a > b$, $(^{+}a) + (^{-}b) = ^{+}(a - b)$

$(^{-}a) + (^{+}b) = ^{-}(a - b)$,

d. If $a < b$, $(^{+}a) + (^{-}b) = ^{-}(b - a)$

$(^{-}a) + (^{+}b) = ^{+}(b - a)$,

e. $(^{+}a) + (^{-}b) = 0$, if $a = b$,

f. $(^{+}a) + 0 = ^{+}a$,

g. $(^{-}a) + 0 = ^{-}a$,

h. $0 + 0 = 0$.

F. Using either rule 1 or rule 5.a from the above section, do the following problems.

Example: $^{+}2 + ^{+}3$

Solution: $^{+}2 + ^{+}3 = ^{+}(2 + 3)$

$= ^{+}5$

Therefore, $^{+}2 + ^{+}3 = ^{+}5$

1. $^{+}2 + ^{+}7$
2. $^{+}7 + ^{+}6$
3. $^{+}7 + ^{+}3$
4. $^{+}3 + ^{+}3$
5. $^{+}2 + ^{+}12$
6. $^{+}11 + ^{+}9$
7. $^{+}10 + ^{+}4$
8. $^{+}5 + ^{+}4$
9. $^{+}6 + ^{+}10$
10. $^{+}20 + ^{+}5$
11. $^{+}8 + ^{+}8$
12. $^{+}11 + ^{+}10$
13. $^{+}2 + ^{+}3$
14. $^{+}3 + ^{+}8$
15. $^{+}3 + ^{+}7$
16. $^{+}20 + ^{+}19$

G. Using either rule 1 or 5.b from section E, do the following problems.

Example: $^{-}2 + {}^{-}3$

Solution: $^{-}2 + {}^{-}3 = {}^{-}(2 + 3)$

$= {}^{-}5$

Therefore, $^{-}2 + {}^{-}3 = {}^{-}5$

1. $^{-}3 + {}^{-}6$
2. $^{-}6 + {}^{-}3$
3. $^{-}3 + {}^{-}5$
4. $^{-}4 + {}^{-}3$
5. $^{-}12 + {}^{-}1$
6. $^{-}1 + {}^{-}4$
7. $^{-}6 + {}^{-}8$
8. $^{-}12 + {}^{-}4$
9. $^{-}11 + {}^{-}11$
10. $^{-}4 + {}^{-}10$
11. $^{-}50 + {}^{-}20$
12. $^{-}19 + {}^{-}20$
13. $^{-}2 + {}^{-}8$
14. $^{-}8 + {}^{-}3$

H. Using either rule 2 or rule 5.e from section E, do the following problems.

Example: $^{+}3 + {}^{-}3$

Solution: $^{+}3 + {}^{-}3 = 0$

1. $^{+}71 + {}^{-}71$
2. $^{-}10 + {}^{+}10$

I. Using either rule 3 or rule 5.c from section E, do the following problems.

Example: $^{+}5 + {}^{-}4$

Solution: $^{+}5 + {}^{-}4 = {}^{+}(5 - 4)$, since $5 > 4$

$= {}^{+}1$

Therefore, $^{+}5 + {}^{-}4 = {}^{+}1$

1. $^{+}4 + {}^{-}3$
2. $^{-}12 + {}^{+}3$
3. $^{-}4 + {}^{+}2$
4. $^{+}4 + {}^{-}2$
5. $^{+}5 + {}^{-}3$
6. $^{-}5 + {}^{+}3$

7. $^{+}8 + {}^{-}6$

8. $^{+}11 + {}^{-}4$

9. $^{-}1000 + {}^{+}1$

10. $^{-}15 + {}^{+}13$

11. $^{+}7 + {}^{-}1$

12. $^{+}10 + {}^{-}6$

13. $^{+}6 + {}^{-}4$

14. $^{+}7 + {}^{-}5$

15. $^{-}20 + {}^{+}19$

16. $^{+}20 + {}^{-}19$

J. Using either rule 3 or rule 5.d from section F, do the following problems.

Example: $^{+}2 + {}^{-}3$

Solution: $^{+}2 + {}^{-}3 = {}^{-}(3 - 2)$, since $2 < 3$

$= {}^{-}1$

Therefore, $^{+}2 + {}^{-}3 = {}^{-}1$

1. $^{+}6 + {}^{-}8$

2. $^{-}5 + {}^{+}8$

3. $^{+}17 + {}^{-}18$

4. $^{-}2 + {}^{+}4$

5. $^{-}5 + {}^{+}6$

6. $^{+}3 + {}^{-}5$

7. $^{-}3 + {}^{+}5$

8. $^{+}4000 + {}^{-}4001$

9. $^{+}3 + {}^{-}4$

10. $^{-}5 + {}^{+}7$

11. $^{+}3 + {}^{-}23$

12. $^{-}2 + {}^{+}6$

13. $^{-}2 + {}^{+}8$

14. $^{-}8 + {}^{+}9$

15. $^{-}7 + {}^{+}11$

16. $^{-}19 + {}^{+}20$

K. Using either rule 4 or rule 5.f, 5.g, 5.h from section F. do the following problems.

1. $^{+}17 + 0$

2. $0 + 0$

3. $0 + {}^{-}4$

4. $^{-}8 + 0$

5. $0 + {}^{+}1$

6. $^{-}1 + 0$

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L. Adding real numbers suggests the idea of taking two trips, as on our imaginary road above, the second being added to the first. You are actually adding real numbers (even though the actual process you use may not be addition in every case) when you use the measures of two trips to find the measure of the direct trip from the starting point of the first trip to the ending point of the second trip.

The same idea is true when you add more than two real numbers. You can think of each real number as the measure of a trip. These can be added together to give the measure of successive trips.

Before you can apply the rules you have practiced to a problem where more than two real numbers are added together, you must note one additional fact.

In a problem such as:

$$^{+}2 + {}^{-}3 + {}^{+}1,$$

where no grouping symbols are given, you should add two of the numbers together to make a single number which can then be added to the other real number in the problem by the rules you have learned.

Example 1: $^{+}2 + {}^{-}3 + {}^{+}1$

Solution 1: $^{+}2 + {}^{-}3 + {}^{+}1 = ({}^{+}2 + {}^{-}3) + {}^{+}1$

$$= {}^{-}1 \quad + {}^{+}1$$

$$= 0$$

Therefore, $^{+}2 + {}^{-}3 + {}^{+}1 = 0$

Example 2: $^{-}2 + {}^{-}1 + {}^{-}3$

Solution 2:
$$\begin{aligned} {}^{-}2 + {}^{-}1 + {}^{-}3 &= ({}^{-}2 + {}^{-}1) + {}^{-}3 \\ &= {}^{-}3 \qquad + {}^{-}3 \\ &= {}^{-}6 \end{aligned}$$

Therefore, $^{-}2 + {}^{-}1 + {}^{-}3 = {}^{-}6$

If grouping symbols are given, you should work the problem in the manner suggested by the symbols.

Example: $({}^{-}5 + {}^{+}3) + ({}^{-}8 + {}^{+}1)$

Solution:
$$\begin{aligned} ({}^{-}5 + {}^{+}3) + ({}^{-}8 + {}^{+}1) &= {}^{-}2 + {}^{-}7 \\ &= {}^{-}9 \end{aligned}$$

Therefore, $({}^{-}5 + {}^{+}3) + ({}^{-}8 + {}^{+}1) = {}^{-}9$

Now do the following problems in the manner you have just been shown.

1. $^{+}2 + {}^{+}2 + {}^{+}4$
2. $^{+}3 + {}^{+}8 + {}^{+}2$
3. $^{-}4 + {}^{-}3 + {}^{-}3$
4. $^{+}10 + {}^{+}20 + {}^{+}10$
5. $^{+}3 + {}^{+}7 + {}^{+}5$
6. $^{+}50 + {}^{+}100 + {}^{+}50$
7. $^{-}500 + {}^{-}500 + {}^{-}1$
8. $^{+}1 + {}^{+}1 + {}^{+}1$
9. $^{+}10 + {}^{-}9 + {}^{+}1$
10. $^{-}11 + {}^{-}1 + {}^{+}2$
11. $^{-}5 + {}^{+}8 + {}^{-}3$
12. $^{+}4 + {}^{+}2 + {}^{+}14$
13. $^{+}10 + {}^{-}10 + {}^{+}1$
14. $^{+}40 + {}^{+}5 + {}^{-}85$
15. $^{+}100 + {}^{+}100 + {}^{-}201$
16. $^{+}3 + {}^{+}7 + {}^{+}5$

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M. This section will give you an opportunity to practice all the rules you have learned. Do all of the following problems.

1. $^{-}2 + {}^{+}3$
2. $^{+}4 + {}^{+}5$
3. $^{-}2 + {}^{+}8$
4. $^{+}4 + {}^{-}3$
5. $^{+}7 + {}^{-}8$
6. $^{+}10 + {}^{+}5$
7. $^{+}9 + {}^{-}3$
8. $^{+}12 + {}^{+}15$
9. $^{+}12 + {}^{+}3$
10. $^{-}2 + {}^{-}5$
11. $^{+}4 + {}^{+}4$
12. $^{+}7 + {}^{-}3$
13. $^{+}7 + {}^{+}3$
14. $^{-}7 + {}^{+}3$
15. $^{-}7 + {}^{-}3$
16. $^{-}2 + {}^{+}2$
17. $^{-}3 + {}^{-}5$
18. $^{-}12 + {}^{+}10$
19. $^{+}2 + {}^{-}1$
20. $^{+}3 + {}^{+}1$
21. $^{-}2 + {}^{+}1$
22. $^{+}6 + {}^{-}7$
23. $^{+}9 + {}^{-}8$
24. $^{-}14 + {}^{+}13$

Here are some that are a little more difficult.

25. $^{+}102 + {}^{-}82$
26. $^{+}750 + {}^{-}350$
27. $^{+}2 + ({}^{-}2 + {}^{+}3)$
28. $^{-}42 + {}^{-}52$
29. $^{+}\frac{1}{2} + {}^{-}\frac{1}{2}$
30. $^{+}190 + {}^{-}81$
31. $^{-}190 + {}^{+}81$
32. $^{+}81 + {}^{-}190$
33. $^{-}7\frac{1}{2} + {}^{+}8\frac{1}{2}$
34. $^{+}3111 + {}^{-}2000$
35. $({}^{-}3 + {}^{+}4) + {}^{+}1$
36. $({}^{+}3 + {}^{-}15) + {}^{+}12$
37. $^{+}2 + ({}^{+}28 + {}^{-}15)$
38. $({}^{-}99 + {}^{-}1) + {}^{+}30$
39. $^{+}87 + ({}^{+}13 + {}^{-}100)$
40. $({}^{+}5 + {}^{+}8) + ({}^{+}1 + {}^{-}12)$

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N. Fill the blanks with numerals which will make the following sentences true.

1. $^{+}14 + {}^{-}14 =$ ____
2. ____ $+ {}^{+}2 = {}^{+}10$
3. $^{-}7 +$ ____ $= {}^{+}2$
4. $^{-}142 + {}^{+}142 + {}^{-}31 =$ ____
5. ____ $+ {}^{-}3 = {}^{+}7$
6. ____ $+ {}^{-}7 = {}^{-}7$
7. $^{+}11 +$ ____ $= 0$
8. $^{-}981 + {}^{+}981 =$ ____
9. $^{-}37 + {}^{+}672 +$ ____ $= {}^{+}672$
10. $^{+}12 +$ ____ $= {}^{+}12$
11. $^{-}4 +$ ____ $= {}^{+}8$
12. ____ $+ {}^{-}2 = {}^{-}4$
13. $^{+}10 +$ ____ $= {}^{+}18$
14. $^{+}70 + {}^{-}20 =$ ____
15. $^{-}3 +$ ____ $= {}^{+}3$
16. $^{+}8 +$ ____ $= {}^{+}8 + {}^{-}971$
17. $^{-}3 + 0 =$ ____
18. ____ $+ {}^{+}1 = {}^{-}1$
19. $^{+}87 +$ ____ $=$ ____ $+ {}^{+}87$
20. $({}^{+}11 + {}^{-}1) + ({}^{+}1 + {}^{-}11) =$ ____
21. $^{-}1000 + {}^{-}1000 =$ ____
22. $^{+}618 + {}^{-}8 =$ ____

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II. Real Number Multiplication--rmE

A. We have seen that real numbers can be added. Since multiplication is merely repeated addition, it can be readily seen that real numbers can also be multiplied. We will teach you how to multiply real numbers by using (1) a rule for multiplying real numbers, and (2) a physical model which may help to clarify the rule.

B. We shall adopt the rule:

> To find the product of two real numbers, we multiply the numerical values of the numbers. If the real numbers have like signs, the product is positive; if they have unlike signs, the product is negative.

1. Here are two examples where real numbers having like signs are multiplied.

    a. ${}^{+}4 \text{ x } {}^{+}2 = {}^{+}8$

    b. ${}^{-}4 \text{ x } {}^{-}2 = {}^{+}8$

    Note that we have chosen a positive number, ${}^{+}8$, for the answer in both cases. This is correct because in example "a" both numbers being multiplied have signs which are alike (like signs). In example "a" both of the signs are " $^{+}$;" hence the sign in the answer is " +."

    In example "b" both numbers have like signs, " $^{-}$ " signs in this case. Therefore, the sign in the answer is " $^{+}$," in accordance with our rule.

2. Here are two examples where real numbers having unlike signs are multiplied.

a. $^{+}4 \times {}^{-}2 = {}^{-}8$

b. ${}^{-}4 \times {}^{+}2 = {}^{-}8$

Note that in both cases the answer was a negative number, ${}^{-}8$. This is correct because in example "a" the numbers being multiplied have unlike signs, one " $^{+}$ " sign and one " $^{-}$ " sign. Therefore, in accordance with our rule, the sign in the answer is " $^{-}$."

In example "b" the two numbers also have unlike signs, again one " $^{-}$ " sign and one " $^{+}$ " sign. Therefore, the sign in the answer is again " $^{-}$."

3. Now go back and read the rule again. If you do not completely understand the rule, ask your teacher to explain it to you further.

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

C. Simplify the following problems.

Example: $^{+}5 \times {}^{-}2$

Solution: $^{+}5 \times {}^{-}2 = {}^{-}10$

1. $^{+}2 \times {}^{+}2$ 2. $^{-}2 \times {}^{+}2$ 3. $^{+}2 \times {}^{-}2$ 4. $^{-}2 \times {}^{-}2$

5. $^{+}8 \times {}^{-}5$ 6. $^{+}8 \times {}^{+}5$ 7. $^{-}8 \times {}^{+}5$ 8. $^{-}8 \times {}^{-}5$

9. $^{+}3 \times {}^{+}3$ 10. $^{+}3 \times {}^{-}3$ 11. $^{-}3 \times {}^{+}3$ 12. $^{-}3 \times {}^{-}3$

13. $^{+}5 \times {}^{+}5$ 14. $^{+}5 \times {}^{-}5$ 15. $^{-}5 \times {}^{+}5$ 16. $^{-}5 \times {}^{-}5$

17. $^{+}10 \times {}^{+}2$ 18. $^{+}10 \times {}^{-}2$ 19. $^{-}10 \times {}^{+}2$ 20. $^{-}10 \times {}^{-}2$

D. Simplify.

1. $^{+}2 \times {}^{+}5$ 2. $^{+}7 \times {}^{+}4$ 3. $^{+}8 \times {}^{-}8$ 4. $^{-}3 \times {}^{-}6$

5. $^{+}6 \times {}^{-}2$ 6. $^{-}2 \times {}^{+}6$ 7. $^{+}5 \times {}^{+}7$ 8. $^{+}8 \times {}^{+}8$

9. $^{+}9 \times {}^{+}10$ 10. $^{+}2 \times {}^{-}10$ 11. $^{-}7 \times {}^{+}8$ 12. $^{+}15 \times {}^{-}3$

13. $^{+}1 \times {}^{-}1$ 14. $^{-}8 \times {}^{+}12$ 15. $^{+}7 \times {}^{-}2$

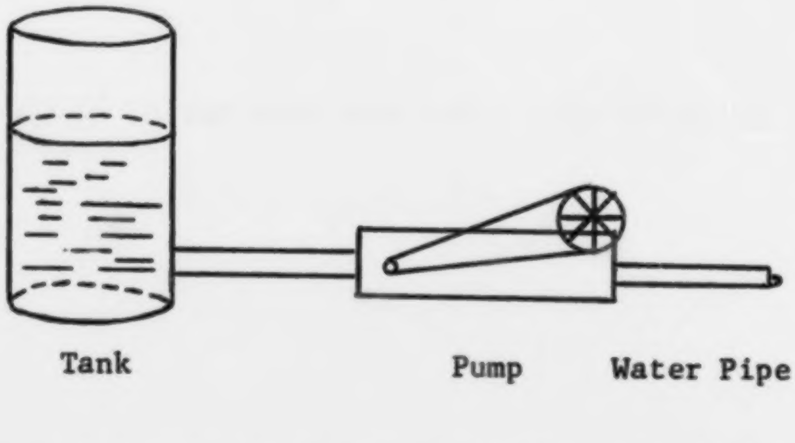
(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

E. We know that real numbers are numbers which measure both distance (amount) and direction. In general, anything which involves an amount and one of two opposite directions can be measured by a real number.

In order to illustrate multiplication of real numbers by a physical model, let us set up a model which contains a pump, a water tank, and a film.

The first part of our model is a tank which contains a certain amount of water. The tank is made of glass, thus permitting you to observe the amount of water (water-volume) in the tank.

A pump is attached to the tank by means of a water pipe. Here is a picture of the situation we have described thus far.



Since the pump can pump in either direction, you need to know whether the flow of water is into the tank or out of the tank.

Suppose that the pump is set to pump water into the tank. The pump is turned on and 2 gallons of water flow through the pipe each minute. At the end of 3 minutes, the pump is turned off. What change in the water-volume (amount of water in the tank) do you see? __________ (Answer: The water-volume would increase [go up] 6 gallons.)

Suppose that the flow is out of the tank. What change do you see in the water-volume? ______ (Answer: The water-volume would decrease [go down] 6 gallons.)

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

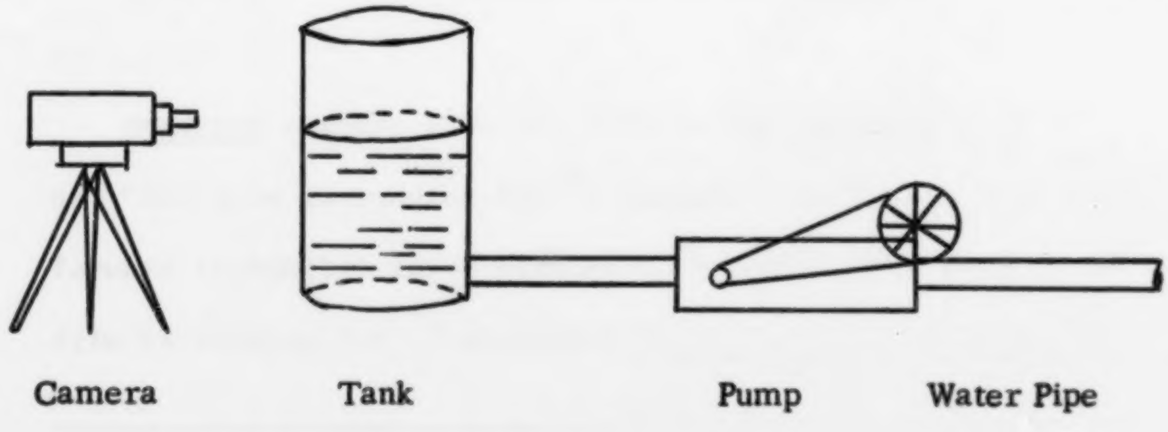
F. Complete the table. We have worked the first problem for you as an example.

	Pump	Time	Observed Change in Volume
1.	4 gal. per minute into the tank	2 min.	Increase of 8 gal.
2.	2 gal. per minute out of the tank	2 min.	
3.	1 gal. per minute OUT	5 min.	
4.	6 gal. per minute IN	3 min.	
5.	3 gal. per minute OUT	4 min.	
6.	10 gal. per minute IN	10 min.	
7.	2 gal. per minute IN	7 min.	
8.	12 gal. per minute OUT	1 min.	

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

G. Let us add to our model a camera which takes a movie of the tank while the pump is operating. Suppose the pump and camera are turned on together, and that 3 gallons of water flow into the tank each minute. After the pump and camera have run for 2 minutes, they are stopped. The film is then developed and projected on a screen. It is possible that the film could be run forward through the projector or backward through the projector.

Suppose the film was run backward through the projector. (Have you ever watched a comedy film in which a man seems to dive up out of the water and land on a diving board, or a race horse runs backward on a muddy track, picking up its footprints as it goes?) If the film were run backward, what change in water-volume would you see on the screen? ________________. (Answer: You would see a decrease of 9 gallons on the screen.)



H. Our pump, tank, film situation involves amounts (the rate of water flow, number of minutes) and direction (filling or emptying the tank, running the film forward or backward) and therefore can be measured by real numbers. We can use real numbers to measure the rate at which the water is being pumped, deciding to use

positive numbers when water flows into the tank,

and

negative numbers when water flows out of the tank.

Thus, if 4 gallons of water are being pumped into the tank each minute, we say that the rate is $^{+}4$ gallons per minute. Explain what is meant by saying that the rate is $^{-}4$ gallons per minute.

____

____

Also, we can use real numbers to measure how long the film is being projected, deciding to use

positive numbers when the film is run forward,

and

negative numbers when the film is run backward.

So, "the film is running for $^{+}3$ minutes," will mean "the film is run forward (normally) for 3 minutes." Explain what is meant by "the film is running for $^{-}3$ minutes." ________________________

____

When we use a numeral such as

"$^{-}4 \times {}^{+}3$"

we will think of $^{-}4$ as measuring the rate at which water is being pumped. We will think of $^{+}3$ as telling us how long the film is being projected. And, finally, we will think of the product

$^{-}4 \times {}^{+}3$

as measuring the change in water-volume which we see on the screen. (Let's agree to use positive numbers to measure observed increases on the screen and negative numbers to measure observed decreases.) In this case, since water is being pumped out of the tank at 4 gallons per minute and the film is being run forward for 3 minutes, the change in water-volume observed on the screen is a decrease of 12 gallons. So, the change is $^{-}12$ gallons. Since we agreed to think of the product as a measure of the change, we can say that

$$^{-}4 \times {}^{+}3 = {}^{-}12.$$

Let's take another case:

$$^{+}9 \times {}^{-}8$$

This number, $^{+}9 \times {}^{-}8$, measures the change in water-volume which you would observe on a screen. The first number, $^{+}9$, measures the rate of pumping (is it filling or is it emptying?); and the second number, $^{-}8$, tells how long the film is being projected (is it being run forward or is it being run backward?). Does $^{+}9 \times {}^{-}8$ measure an observed increase in water-volume or an observed decrease? ______________________________

Since $^{+}9$ would indicate that 9 gallons of water are being pumped into the tank each minute, and $^{-}8$ would indicate the film being run backward for 8 minutes, we would observe a decrease in water volume of 72 gallons. Therefore, we can say that

$$^{+}9 \times {}^{-}8 = {}^{-}72$$

Suppose the water was being pumped out of the tank, for $^{+}2$ minutes, at the rate of 5 gallons per minute ($^{-}5$). If the film were run forward through the projecter $(^{+})$, what change in water-volume would you observe on the screen? ______________________ (Answer: 10 gal. decrease, or $^{-}10$). If the film were run backward $(^{-})$, what change would you see on the screen? ____________ (Answer: 10 gal. increase, or $^{+}10$).

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

I. Complete the table below. We have solved the first problem for you as an example.

	Pump	Movie	Observed Change in Volume
1.	$^{+}5$	$^{-}3$	$^{-}15$
	Filling 5 gal. per minute	Running backward 3 minutes	Decrease of 15 gal.

In problem 1, you are told that the pump is filling (forcing water INTO) the tank at the rate of 5 gallons per minute. From the second column you learn that the movie has been run backward for 3 minutes. Since the pump is filling the tank, and since the film is run backward, the volume of water appears to be decreasing (just the opposite of what actually happened in the tank). So we see on the screen a decrease in volume of 15 gallons, which we write in the box provided.

	Pump	Movie	Observed Change in Volume
2.	$^{+}1$	$^{+}13$	
	Filling 1 gal. per minute	Running forward 13 minutes	
3.	$^{-}4$	$^{+}2$	
	Emptying 4 gal. per minute	Running forward 2 minutes	

	Pump	Movie	Observed Change in Volume
4.	$^{+}10$	$^{-}5$	
	Filling 10 gal. per minute	Running backward 5 minutes	
5.	$^{-}3$	$^{-}4$	
	Emptying 3 gal. per minute	Running backward 4 minutes	
6.	$^{-}11$	$^{+}3$	
	Emptying 11 gal. per minute	Running forward 3 minutes	
7.	$^{+}2$	$^{+}6$	
	Filling 2 gal. per minute	Running forward 6 minutes	
8.	$^{+}5$	$^{-}2$	
	Filling 5 gal. per minute	Running backward 2 minutes	
9.	$^{-}5$	$^{+}2$	
	Emptying 5 gal. per minute	Running forward 2 minutes	

	Pump	Movie	Observed Change in Volume
10.	$^{+}5$	$^{+}2$	
	Filling 5 gal. per minute	Running forward 2 minutes	
11.	$^{-}5$	$^{-}2$	
	Emptying 5 gal. per minute	Running backward 2 minutes	

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

J. In the following problems, find the real number which measures the observed change in volume. You do not need to write out in words what observed change you would see on the screen, but you may do so if you think it will help you. We have solved the first problem for you as a sample.

	Pump	Movie	Observed Change in Volume
1.	$^{+}1$	$^{-}17$	$^{-}17$
	Filling 1 gal. per minute	Running backward 17 minutes	
2.	$^{+}11$	$^{+}2$	
	Filling 11 gal. per minute	Running forward 2 minutes	
3.	$^{-}2$	$^{+}8$	
	Emptying 2 gal. per minute	Running forward 8 minutes	
4.	$^{+}3$	$^{-}2$	
	Filling 3 gal. per minute	Running backward 2 minutes	
5.	$^{-}4$	$^{-}5$	
	Emptying 4 gal. per minute	Running backward 5 minutes	

	Pump	Movie	Observed Change in Volume
6.	$^{-}6$ Emptying 6 gal. per minute	$^{+}3$ Running forward 3 minutes	
7.	$^{+}6$ Filling 6 gal. per minute	$^{+}10$ Running forward 10 minutes	
8.	$^{+}7$ Filling 7 gal. per minute	$^{-}2$ Running backward 2 minutes	
9.	$^{-}100$ Emptying 100 gal. per minute	$^{-}9$ Running backward 9 minutes	
10.	$^{-}6$ Emptying 6 gal. per minute	$^{+}6$ Running forward 6 minutes	

(Note: In the rest of the problems you are given real numbers; you should find the real number which should fill the third column. You may fill the other spaces if you wish. Problem 11 has been worked as a sample for you.

11.	$^{+}4$	$^{+}3$	$^{+}12$
12.	$^{-}7$	$^{+}2$	
13.	$^{-}11$	$^{-}5$	
14.	$^{+}10$	$^{-}6$	
15.	$^{+}5$	$^{-}4$	
16.	$^{-}5$	$^{+}4$	
17.	$^{+}5$	$^{+}4$	

18.

$^{-}5$	$^{-}4$	

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

K. The table below contains problems dealing with the pump-tank-movie interpretation. From each problem you can learn how to multiply a pair of real numbers. We have solved the first problem as a sample for you, with all columns filled out. You do not need to complete the "pump" and "movie" columns but may do so if you find it helpful.

	Pump	Movie	Observed Change in Volume
1.	Filling 4 gal. per min	Running backward 2 min.	Decrease of 8 gallons
	$^{+}4$	$^{-}2$	$^{-}8$
	Corresponding multiplication statement: $^{+}4 \times {}^{-}2 = {}^{-}8$		
2.			
	$^{-}4$	$^{+}2$	
	Corresponding multiplication statement:		
3.			
	$^{+}4$	$^{+}2$	
	Corresponding multiplication statement:		

	Pump	Movie	Observed Change in Volume
4.			
	$^{-}4$	$^{-}2$	
	Corresponding multiplication statement		
5.			
	$^{+}8$	$^{+}3$	
	Corresponding multiplication statement:		
6.			
	$^{+}3$	$^{-}8$	
	Corresponding multiplication statement:		
7.			
	$^{-}8$	$^{-}3$	
	Corresponding multiplication statement:		

	Pump	Movie	Observed Change in Volume
8.			
	$^{-}5$	$^{-}6$	
	Corresponding multiplication statement: $^{-}5 \times {}^{-}6 =$		
9.			
	$^{+}7$	$^{-}3$	
	Corresponding multiplication statement: $^{+}7 \times {}^{-}3 =$		
10.			
	$^{-}8$	0	
	Corresponding multiplication statement: $^{-}8 \times 0 =$		
11.			
	$^{-}6$	$^{-}4$	
	Corresponding multiplication statement: $^{-}6 \times {}^{-}4 =$		

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

L. Do the following problems. You may use the rule you learned on page A2.16, or you may use the pump-tank-film model.

1. ${}^{-}6 \times {}^{-}4$
2. ${}^{-}100 \times {}^{-}2$
3. ${}^{-}5 \times {}^{-}10$
4. ${}^{-}3 \times {}^{-}15$
5. $0 \times {}^{-}7$
6. ${}^{+}6 \times 0$
7. 0×0
8. ${}^{-}12 \times 0$
9. ${}^{-}4 \times {}^{-}1$
10. ${}^{-}7 \times {}^{+}1$
11. ${}^{-}3 \times {}^{-}8$
12. ${}^{+}6 \times {}^{-}5$
13. ${}^{-}3 \times {}^{-}2$
14. ${}^{-}8 \times {}^{+}2$
15. ${}^{+}7 \times {}^{-}8$
16. ${}^{+}7 \times {}^{-}6$
17. ${}^{+}75 \times {}^{+}5$
18. ${}^{-}86 \times {}^{-}10$

The following problems should pose no problem, but just in case, here is an example worked for you:

$$({}^{-}3 \times {}^{+}4) \times {}^{+}5$$

$$\underbrace{{}^{-}12}_{} \quad \times {}^{+}5 = {}^{-}60$$

19. $({}^{+}2 \times {}^{-}3) \times {}^{-}4$
20. $({}^{-}2 \times {}^{+}7) \times {}^{-}8$
21. $({}^{+}5 \times {}^{-}3) \times {}^{-}4$
22. $({}^{+}6 \times {}^{-}2) \times {}^{-}3$
23. ${}^{+}4 \times ({}^{-}3 \times {}^{-}7)$
24. ${}^{-}6 \times ({}^{+}2 \times {}^{-}5)$
25. $({}^{+}73 \times {}^{-}81) \times 0$
26. $({}^{+}5 \times {}^{-}17) \times ({}^{-}3 \times 0)$
27. ${}^{-}981 \times 0$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

M. Simplify. Be careful not to confuse addition signs with multiplication signs.

1. $(^{+}5 + {}^{-}3) \times {}^{-}7$
2. $(^{+}3 \times {}^{-}4) + {}^{-}6$
3. $(^{+}8 \times {}^{-}3) + {}^{-}5$
4. $(^{+}12 + {}^{-}11) \times {}^{-}3$
5. $(^{+}1 + {}^{+}1) + {}^{+}1$
6. $(^{+}1 \times {}^{+}1) + {}^{+}1$
7. $(^{-}1 + {}^{-}1) + {}^{-}1$
8. $(^{-}1 \times {}^{-}1) \times {}^{-}1$
9. $(^{-}4 \times {}^{-}2) + (^{-}5 \times {}^{+}6)$
10. $(^{-}3 \times {}^{+}7) + (^{+}8 \times {}^{-}4)$
11. $(^{+}71 + {}^{-}11) \times (^{+}6 + {}^{-}4)$
12. $(^{-}8 + 0) \times (^{+}4 \times {}^{+}2)$

N. Fill in the blanks to make true sentences.

1. $^{+}5 \times$ ____ $= {}^{-}20$
2. $^{-}3 \times$ ____ $= {}^{+}18$
3. $^{+}7 \times$ ____ $= {}^{+}21$
4. $^{-}7 \times$ ____ $= {}^{-}21$
5. ____ $+ {}^{-}3 = {}^{+}9$
6. ____ $\times {}^{-}3 = {}^{+}9$
7. $^{+}8 +$ ____ $= {}^{-}8$
8. $^{+}8 \times$ ____ $= {}^{-}8$
9. $^{+}3 \times$ ____ $= {}^{+}1$
10. $^{+}3 +$ ____ $= 0$
11. $^{+}3 \times$ ____ $= 0$
12. $^{+}3 +$ ____ $= {}^{+}1$
13. $^{+}8 \times {}^{-}2 = {}^{-}(8 \times$ ____$)$
14. $^{-}5 \times$ ____ $= {}^{+}(5 \times 2)$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

III. Distributive Principle for Multiplication Over Addition--dpE

A. There is a principle which we have not yet studied which works with all numbers of arithmetic. This is called the distributive principle for multiplication over addition. We shall refer to it as the "dpma" for short. This principle tells you that:

> The product of a first number times the sum of a second and a third number is equal to the product of the first and second plus the product of the first and third.

Stated another way:

> Let a, b, and c be any numbers; then
>
> $$a \times (b + c) = (a \times b) + (a \times c)$$

Here is an example where we can see a use for the dpma.

Example 1: $9 \times (10 + 2)$

Solution: If we compute

$$9 \times (10 + 2)$$

we find that

$$9 \times (10 + 2) = 9 \times 12.$$

Nine multiplied by twelve is a fact that most of us have not committed to memory. Therefore, most of us are forced to go through the mechanical process of multiplying, carrying, re-multiplying, etc. Finally, we come up with one hundred and eight as the answer.

$$\begin{aligned} 9 \times (10 + 2) &= 9 \times 12 \\ &= 108 \end{aligned}$$

Therefore, $9 \times (10 + 2) = 108$

Let us now use the dpma to make the solution to this problem easier. By replacing "a" with "9," "b" with "10," and "c" with "2" in the pattern

$$a \times (b + c) = (a \times b) + (a \times c)$$

we come up with

$$9 \times (10 + 2) = (9 \times 10) + (9 \times 2).$$

Now let us compute

$$(9 \times 10) + (9 \times 2).$$

We find that

$$\begin{aligned}(9 \times 10) + (9 \times 2) &= 90 + 18 \\ &= 108\end{aligned}$$

Most of us can perform these processes mentally, as most of us have committed to memory the facts 9 x 10 and 9 x 2, and can readily add 90 to 18 mentally. Therefore, we see that the dpma can help us to solve some problems more easily.

Let us try another example.

Example 2: (15 x 7) + (15 x 3)

Solution: If we compute we find that

$$\begin{aligned}(15 \times 7) + (15 \times 3) &= 105 + 45 \\ &= 150\end{aligned}$$

Fifteen multiplied by seven is a fact that most of us have not committed to memory. Therefore, most of us are forced to go through a mechanical computing process in order to arrive at the answer, one hundred and five. The same may be true of the second fact, fifteen multiplied by three.

Let us now use the dpma to make the solution to this problem easier. By replacing "a" with "15," "b" with "7," and "c" with "3" in the pattern

$$(a \times b) + (a \times c) = a \times (b + c)$$

we come up with

$$(15 \times 7) + (15 \times 3) = 15 \times (7 + 3).$$

Now let us compute

$$15 \times (7 + 3).$$

We find that

$$\begin{aligned} 15 \times (7 + 3) &= 15 \times 10 \\ &= 150. \end{aligned}$$

Most of us can mentally multiply 15 x 10. Therefore, we see that the dpma can help us to solve some problems more easily.

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

B. Here are some practice problems. The first few are completed for you. You should read through them and then complete the problems which have been left for you to do. Remember to use the dpma on each problem. You should also remember our agreement that when grouping symbols are not used, multiplication takes priority over addition.

Example: 7 x 12 + 3 x 12

Solution: (7 x 12) + (3 x 12) would be the correct way to use the grouping symbols. Of course, by using the dpma you would rewrite the problem as:

$$\begin{aligned}(7 \times 12) + (3 \times 12) &= (7 + 3) \times 12 \\ &= 10 \times 12 \\ &= 120\end{aligned}$$

Ask your teacher for help if you have any difficulty.

1. 4 x 11 + 6 x 11 = (4 + 6) x 11 = 10 x 11 = 110
2. 8 x 23 + 2 x 23 = (8 + 2) x 23 = 10 x 23 = 230
3. 5 x 9 + 6 x 9 = (5 + 6) x 9 = 11 x 9 = _____
4. 2 x 17 + 8 x 17 = (2 + _____) x 17 = 10 x 17 = _____
5. 17 x 15 + 3 x 15 = (_____ + 3) x 15 = 20 x _____ = _____
6. 9 x 4 + 21 x 4 = (9 + 21) x _____ = _____ x _____ = _____
7. 91 x 8 + 9 x 8 = (_____ + _____) x 8 = _____ x _____ = _____
8. $\frac{2}{3}$ x 12 + $\frac{1}{3}$ x 12 = (_____ + _____) x _____ = _____ x _____ = _____
9. 7 x 6 + 4 x 6 = (_____ + _____) x _____ = _____ x _____ = _____
10. 8 x 437 + 2 x 437 = (_____ + _____) x _____ = _____ x _____ = _____

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

C. Using the dpma, solve the following problems. Show all your computation in the space at the right.

1. $67 \times 25 + 33 \times 25 =$ _____

2. $11 \times 2\frac{1}{4} + 11 \times 2\frac{3}{4} =$ _____

Have your teacher check your work to see if you used the dpma.

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

D. Simplify, using the dpma.

Example: $6 \times 5 + 14 \times 5$

Solution: $6 \times 5 + 14 \times 5 = (6 + 14) \times 5$

$= 20 \times 5$

$= 100$

1. $3 \times 9 + 8 \times 9$
2. $8 \times 83 + 8 \times 17$
3. $13 \times 5 + 15 \times 5$
4. $1 \times 6 + 99 \times 6$
5. $5 \times 987 + 5 \times 13$
6. $88 \times 21 + 12 \times 21$
7. $7 \times 4 + 7 \times 7$
8. $4 \times 19 + 4 \times 6$
9. $29 \times 3 + 4 \times 3$
10. $\frac{1}{5} \times 15 + \frac{4}{5} \times 15$
11. $3 \times 12 + 2 \times 12$
12. $6 \times 14 + 4 \times 14$
13. $3 \times 5 + 3 \times 4$
14. $3 \times 7 + 5 \times 7$
15. $3 \times 51 + 17 \times 51$
16. $47 \times 2 + 47 \times 8$
17. $12 \times 11 + 12 \times 1$
18. $11 \times 73 + 9 \times 73$
19. $138 \times 2 + 138 \times 8$
20. $29 \times 3 + 21 \times 3$
21. $4 \times 97 + 6 \times 97$
22. $13 \times 19 + 7 \times 19$
23. $19 \times 13 + 19 \times 7$
24. $163 \times 5 + 163 \times 5$
25. $163 \times 6 + 163 \times 4$
26. $9 \times 163 + 1 \times 163$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

E. The first few of the following problems are completed for you. Read through them and complete the problems which have been left for you to do. Remember to use the dpma on each problem.

1. 8 x 3 + 8 x 4 = 8 x 7 = 56
2. 7 x 6 + 7 x 4 = 7 x 10 = 70
3. 9 x 8 + 9 x 2 = 9 x 10 = 90
4. 4 x 9 + 4 x 1 = 4 x ____ = ____
5. 5 x 13 + 5 x 7 = ____ x ____ = ____
6. 6 x 49 + 6 x 51 = ____ x ____ = ____
7. 18 x 3 + 7 x 3 = 25 x ____ = ____
8. 1 x 8 + 9 x 8 =
9. 2 x 8 + 8 x 8 =
10. 3 x 8 + 7 x 8 =
11. 4 x 8 + 6 x 8 =
12. 5 x 8 + 5 x 8 =
13. 2 x 8 + 6 x 8 =

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

F. Fill in the frames to make true sentences.

Example: 2 x 6 + 8 x 6 = (2 + 8) x □

Solution: Using the dpma, we know that:

2 x 6 + 8 x 6 = (2 + 8) x 6.

Therefore, the frame could be filled with a '6' to make the sentence true.

1. 7 x 8 + 3 x 8 = (7 + □) x 8
2. 5 x 6 + 3 x 6 = (5 + 3) x △
3. ⬡ x (7 + 8) = ⬡ x 7 + ⬡ x 8
4. 8 x 5 + 2 x 5 = (□ + 2) x 5
5. (6 + 4) x 5 = (△ + 2) x 5
6. 7 x (3 + 2) = 7 x 3 + 7 x □

(Remember our agreement about sentences with two or more frames of the same shape. Each frame must be filled with a numeral for the same number.)

7. 5 x (4 + 8) = □ x 4 + □ x 8
8. 6 x 9 + △ x 9 = (6 + △) x 9
9. 3 x 16 + 7 x 16 = ⬡ x 16
10. 5 x 19 + 5 x 19 = (5 + 5) x △
11. 11 x 4 + 11 x □ = 11 x 7
12. 53 x ⬡ + 53 x 17 = 53 x (13 + 17)
13. △ x 9 + △ x 11 = 132 x (9 + 11)
14. 87 x (93 + □) = 87 x 93 + 87 x □
15. 561 x 432 + 413 x 432 = (△ + 413) x 432

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

G. In each sentence fill the frames with numerals which will make the sentence true.

Remember when a frame of one shape appears more than once in an expression it must be filled by a numeral for the same number each time. For example, in the sentence: $\square + \square = 14$, $\boxed{7} + \boxed{7} = 14$ is allowed, but $\boxed{6} + \boxed{8} = 14$ is not allowed.

Example 1: $6 \times \square + 4 \times \square = 10$

Solution 1: Using the dpma we see that:

$$6 \times \square + 4 \times \square = 10$$

becomes

$$(6 + 4) \times \square = 10$$
$$10 \times \square = 10.$$

Therefore, we can readily see that if we fill the frame with '1' we will make the sentence true.

Example 2: $3 \times \square + 7 \times \square = 80$

Solution 2: Using the dpma we see that

$$3 \times \square + 7 \times \square = 80$$

becomes

$$(3 + 7) \times \square = 80$$
$$10 \times \square = 80.$$

Therefore, if we fill the frame with '8' we will make the sentence true.

Example 3: $6 \times \square + 4 \times \square = 22$

Solution 3: $6 \times \square + 4 \times \square = 22$

becomes

$$(6 + 4) \times \square = 22$$

$$10 \times \square = 22.$$

The frame should be filled with '2.2.'

Example 4: $5 \times \square + 7 \times \square = 48$

Solution 4: $5 \times \square + 7 \times \square = 48$

becomes

$$(5 + 7) \times \square = 48$$

$$12 \times \square = 48.$$

The frame should be filled with '4.'

1. $6 \times \square + 4 \times \square = 30$
2. $6 \times \square + 4 \times \square = 60$
3. $6 \times \square + 4 \times \square = 90$
4. $6 \times \square + 4 \times \square = 110$
5. $6 \times \square + 4 \times \square = 250$
6. $6 \times \square + 4 \times \square = 20$
7. $6 \times \square + 4 \times \square = 1000$
8. $6 \times \square + 4 \times \square = 311$
9. $3 \times \square + 7 \times \square = 50$
10. $8 \times \square + 2 \times \square = 70$
11. $98 \times \square + 2 \times \square = 200$
12. $4 \times \square + 7 \times \square = 22$
13. $6 \times \square + 3 \times \square = 90$
14. $87 \times \square + 13 \times \square = 500$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

H. Solve the following problems. You will find that the dpma helps you with most of them.

1. 3 x 6
2. 21 + 3
3. 13 + 1
4. 5 x 7
5. 4 x 8 + 3
6. 3 x 5 + 4 x 5
7. 8 x 17 + 2 x 17
8. 12 x 4 + 12 x 16
9. 2 x 21 + 6 x 21
10. 91 x 7 + 9 x 7
11. 374 x 2 + 374 x 2
12. 6 x 1 + 6 x 199
13. 7 x 6 + 4 x 6
14. 8 x 6 + 12 x 6
15. 3 x 13 + 7 x 13
16. 6 x 7 + 7 x 7
17. 4 x 22 + 10 x 22
18. 6 x 3 + 6 x 7
19. 8 x 7 + 2 x 7
20. 7 x 5 + 4 x 5
21. 5 x 3 + 5 x 8
22. 3 x 88 + 3 x 12
23. 1 x 4 + 999 x 4
24. 14 x 8 + 6 x 8
25. 7 x 5 + 7 x 4
26. 602 x 7 + 398 x 7
27. 4 x 14 + 4 x 11
28. 52 x 9 + 48 x 9
29. 97 x 6 + 97 x 4
30. 19 x 13 + 19 x 87

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(This page should be used only after teacher presentation of the story of "Distributive Ben Harris," adapted from Wirtz, Robert W., Sawyer, W. W., and Botel, Morton, Math Workshop for Children, Level D. Encyclopaedia Britannica Press, Inc., 1962, pp. 20-22.)

I. Remember that Ben did not know any multiplication fact in which either factor is greater than 5. You must imagine you have the same limitation while doing these problems.

D P M A

1. $2 \times 9 = 2 \times (\underline{5} + \underline{4}) = (2 \times 5) + (2 \times 4) = \underline{10} + \underline{8} = \_\_$

2. $10 \times 4 = (\underline{5} + \_\_) \times \underline{4} = (5 \times 4) + (\_\_ \times 4) = \underline{20} + \_\_ = \_\_$

3. $3 \times 7 = \_\_ \times (\_\_ + \_\_) = (\_\_ \times \_\_) + (\_\_ \times \_\_) = \_\_ + \_\_ = \_\_$

4. $8 \times 3 = (\_\_ + \_\_) \times \_\_ = (\_\_ \times \_\_) + (\_\_ \times \_\_) = \_\_ + \_\_ = \_\_$

5. $5 \times 6 = \_\_ \times (\_\_ + \_\_) = (\_\_ \times \_\_) + (\_\_ \times \_\_) = \_\_ + \_\_ = \_\_$

6. $2 \times 10 = \_\_ \times (\_\_ + \_\_) = (\_\_ \times \_\_) + (\_\_ \times \_\_) = \_\_ + \_\_ = \_\_$

7. $6 \times 3 = (\_\_ + \_\_) \times \_\_ = (\_\_ \times \_\_) + (\_\_ \times \_\_) = \_\_ + \_\_ = \_\_$

8. $2 \times 7 = \_\_ \times (\_\_ + \_\_) = (\_\_ \times \_\_) + (\_\_ \times \_\_) = \_\_ + \_\_ = \_\_$

9. $9 \times 4 = (\_\_ + \_\_) \times \_\_ = (\_\_ \times \_\_) + (\_\_ \times \_\_) = \_\_ + \_\_ = \_\_$

10. $3 \times 10 = \_\_ \times (\_\_ + \_\_) = (\_\_ \times \_\_) + (\_\_ \times \_\_) = \_\_ + \_\_ = \_\_$

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J. On the next page you have an opportunity to use the dpma in a way similar to the way Ben Harris used it. If you are given a problem that looks like this:

$$9 \times 8 =$$

$$9 \times 9 =$$

$$\boxed{9 \times 17} =$$

and are told that you only need to do the problem in the box, then you have two choices. One way to approach the problem is to multiply 9 x 17. Since most of us have not committed that fact to memory, we have to go through the mechanical process of multiplying, carrying, etc.

An easier way to find the answer to 9 x 17 would be to rename 17 and then apply the dpma. Let us rename 17 with 8 + 9. Then our problem becomes:

$$9 \times (8 + 9)$$

By applying the dpma we find

$$\begin{aligned} 9 \times (8 + 9) &= (9 \times 8) + (9 \times 9) \\ &= 72 \quad + \quad 81 \\ &= 153 \end{aligned}$$

Most of us have memorized the facts 9 x 8 and 9 x 9 and therefore find this second approach easier than the first. Applying this to the problem we first stated, we see that:

$$9 \times 8 = 72$$

$$9 \times 9 = \underline{81}$$

$$9 \times 17 = 153$$

On this page you only need to do the problems in the boxes. You may find it easier to do all the problems and use the dpma as shown on the previous page.

2 x 4 =	5 x 5 =	6 x 8 =
2 x 8 =	5 x 10 =	6 x 10 =
2 x 12 =	5 x 15 =	6 x 18 =
7 x 2 =	3 x 9 =	4 x 3 =
7 x 10 =	3 x 10 =	4 x 20 =
7 x 12 =	3 x 19 =	4 x 23 =

4 x3	10 x3	14 x3	6 x8	10 x8	16 x8	2 x7	10 x7	12 x7
9 x6	40 x6	49 x6	5 x5	70 x5	75 x5	6 x7	20 x7	26 x7
5 x9	50 x9	55 x9	8 x4	20 x4	28 x4	5 x3	30 x3	35 x3
9 x2	60 x2	69 x2	10 x4	11 x4	21 x4	10 x8	9 x8	19 x8

4 x 9 =	7 x 5 =	9 x 3 =
4 x 40 =	7 x = 420	9 x = 450
4 x 49 =	7 x = 455	9 x = 477
7 x 3 =	8 x 8 =	4 x = 28
7 x 70 =	8 x 70 =	4 x = 280
7 x 73 =	8 x 78 =	4 x = 308

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K. Complete these examples as rapidly and accurately as you can. Use the dpma to solve the problems in the boxes if you find it easier.

3 x6	50 x6	53 x6	4 x7	50 x7	54 x7	3 x9	40 x9	43 x9
9 x4	60 x4	69 x4	6 x6	40 x6	46 x6	5 x7	60 x7	65 x7
4 x5	40 x5	44 x5	9 x5	80 x5	89 x5	8 x9	70 x9	78 x9
6 x4	90 x4	96 x4	9 x8	70 x8	79 x8	7 x7	90 x7	97 x7
9 x9	90 x9	99 x9	5 x4	80 x4	85 x4	6 x8	90 x8	96 x8
8 x2	70 x2	78 x2	3 x8	80 x8	83 x8	5 x6	80 x6	85 x6
8 x8	50 x8	58 x8	6 x3	80 x3	86 x3	7 x9	50 x9	57 x9
4 x6	70 x6	74 x6	3 x5	90 x5	93 x5	8 x5	50 x5	58 x5
7 x3	90 x3	97 x3	6 x9	80 x9	86 x9	8 x3	60 x3	68 x3
4 x4	60 x4	64 x4	2 x6	90 x6	92 x6	9 x7	80 x7	89 x7

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L. You will find that it is easier to work the examples in the boxes, using the dpma, after you have worked the other examples.

3 x 2 =	6 x 3 =	4 x 5 =
3 x 10 =	6 x 20 =	4 x 20 =
3 x 300 =	6 x 100 =	4 x 300 =
3 x 312 =	6 x 123 =	4 x 325 =

5 x5	20 x5	300 x5	325 x5	7 x9	00 x9	400 x9	407 x9
7 x2	10 x2	500 x2	517 x2	0 x7	90 x7	800 x7	890 x7
8 x5	40 x5	700 x5	748 x5	9 x6	40 x6	700 x6	749 x6
6 x4	40 x4	800 x4	846 x4	1 x8	90 x8	200 x8	291 x8
8 x6	50 x6	100 x6	158 x6	4 x7	00 x7	600 x7	604 x7
9 x3	40 x3	800 x3	849 x3	7 x8	40 x8	600 x8	647 x8
6 x5	50 x5	200 x5	256 x5	0 x9	10 x9	200 x9	210 x9
8 x8	50 x8	300 x8	358 x8	9 x9	50 x9	300 x9	359 x9

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M. Complete this page.

9 x4	60 x4	69 x4	6 x3	70 x3	76 x3	5 x7	90 x7	95 x7
4 x5	40 x5	44 x5	7 x2	80 x2	87 x2	9 x3	10 x3	19 x3
6 x4	90 x4	96 x4	9 x9	90 x9	99 x9	8 x7	50 x7	58 x7
6 x5	70 x5	76 x5	4 x4	40 x4	44 x4	10 x8	7 x8	17 x8
7 x7	30 x7	37 x7	20 x6	6 x6	26 x6	9 x8	50 x8	59 x8
3 x7	20 x7	23 x7	9 x5	100 x5	109 x5	40 x10	18 x10	58 x10

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

N. Work the examples in the heavy outlined boxes. You do not need to do the others, but you may do them first and then use the dpma.

2 x 14 = 28	5 x 16 =	3 x 21 =
10 x 14 = 140	10 x 16 =	20 x 21 =
12 x 14 =	15 x 16 =	23 x 21 =
4 x 12 =	6 x 15 =	1 x 23 =
10 x 12 =	10 x 15 =	20 x 23 =
14 x 12 =	16 x 15 =	21 x 23 =

42 x9	42 x10	42 x19	48 x7	48 x20	48 x27

2 x 19 =	4 x 25 =	2 x 35 =
40 x 19 =	30 x 25 =	60 x 35 =
42 x 19 =	34 x 25 =	62 x 35 =

34 x5	34 x20	34 x25	62 x5	62 x30	62 x35

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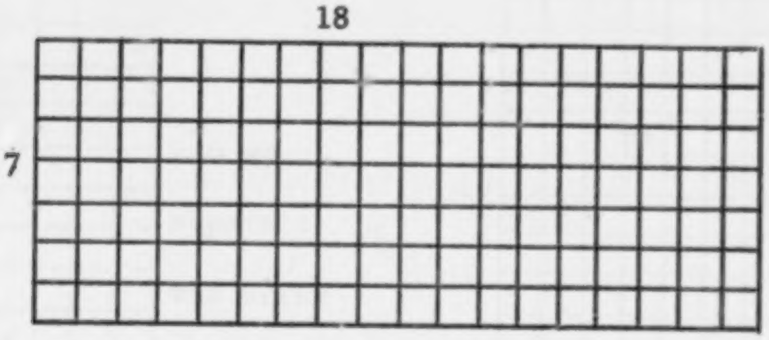
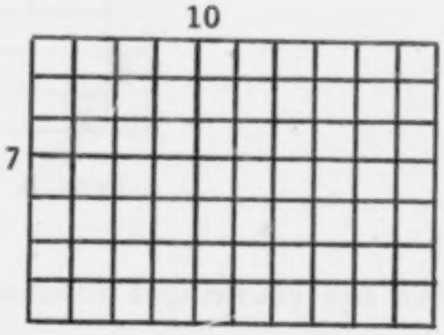
O. **Distributive property of multiplication over addition.**

To find the product of 7 and 18, think of a 7 by 18 array.



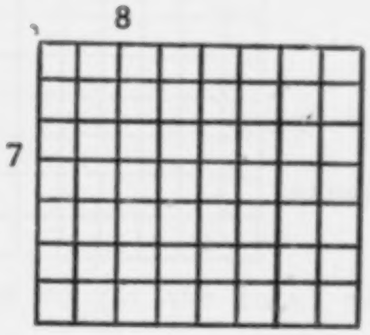
Separate it into two arrays showing products you already know.

For example:



7 by 10 array

$7 \times 10 = 70$



7 by 8 array

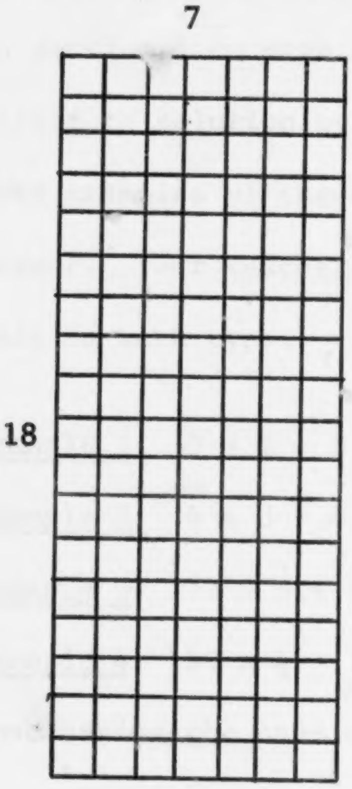
$7 \times 8 = 56$

These arrays help us see that

$$\begin{aligned} 7 \times 18 &= 7 \times (10 + 8) \\ &= (7 \times 10) + (7 \times 8) \\ &= 70 + 56 \\ &= 126. \end{aligned}$$

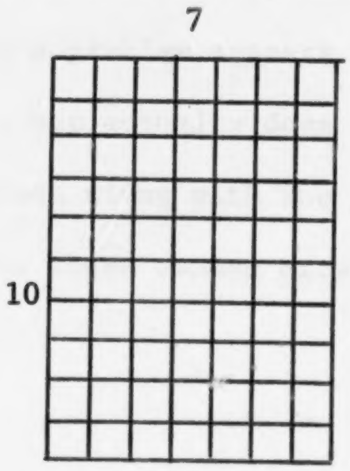
When we write $(7 \times 10) + (7 \times 8)$ in place of $7 \times (10 + 8)$, we are using the distributive property of multiplication over addition.

Now, suppose we find the product of 18 and 7.

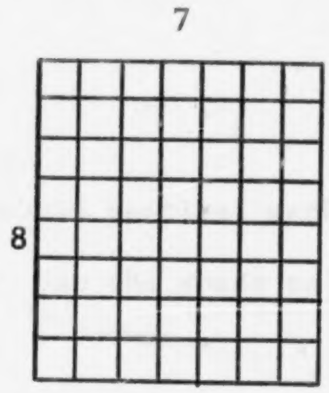


(18 by 7 array)

Can we separate the array like this?



(10 by 7 array)



(8 by 7 array)

Find the products separately and add them to get the total number of elements in the 18 by 7 array.

$$\begin{aligned} 18 \times 7 &= (10 + 8) \times 7 \\ &= (10 \times 7) + (8 \times 7) \\ &= 70 + 56 \\ &= 126 \end{aligned}$$

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P. Remember that the dpma works only when it fits the pattern shown in section A on page A2.38. Sometimes a problem appears to lend itself to solution by use of the dpma, but actually does not fit. Some examples of these problems are given along with the incorrect answer. Your teacher will show you how these common errors are made in each type.

Example 1: 5 x 2 + 8 x 5 = 100

Example 2: 4 x 3 + 4 x 7 = 80

Example 3: 11 x 6 + 4 x 4 = 150

Example 4: 90 x $\frac{1}{2}$ + 10 x $\frac{1}{2}$ = 100

Q. Remembering the caution given in the above section, mark each of the following sentences true or false. Use the space on the right for any computations.

1. 18 x 4 + 7 x 4 = 100
2. 6 x 7 + 3 x 6 = 120
3. 5 x 2 + 5 x 8 = 100
4. 82 x 6 + 17 x 6 = 600
5. 5 x 49 + 5 x 51 = 500
6. 9 x 3 + 4 x 7 = 130
7. $\frac{1}{5}$ x 91 + $\frac{4}{5}$ x 91 = 91
8. 5 x 16 + 5 x 4 = 100
9. 99 x $\frac{1}{3}$ + 1 x $\frac{2}{3}$ = 100
10. 3 x 7 + 7 x 3 = 100
11. 91 x 7 + 8 x 7 = 700

IV. Introducing Exponents--enE

A. Product expressions. The sentence 5 x 6 = 30 shows that 30 is the product of 5 and 6. Because it names a number as a product, an expression like 5 x 6 is called a product expression. Other product expressions for 30 are 10 x 3, 2 x 15, 30 x 1, and 2 x 3 x 5.

B. Exercises. Write one or more product expressions for each of the following numerals.

Example: 15 = 3 x 5

1. 18
2. 24
3. 20
4. 32
5. 25
6. 12

Write a simpler numeral for the number named by each of the following product expressions.

Example: 4 x 7 = 28

7. 3 x 9
8. 4 x 10
9. 3 x 15
10. 4 x 16
11. 6 x 30
12. 5 x 31
13. 2 x 4 x 4
14. 4 x 2 x 2
15. 5 x 5 x 5
16. 2 x 2 x 2 x 2 x 3
17. 2 x 5 x 5
18. 4 x 3 x 2 x 2

C. Repeated factors. The sentence 5 x 6 = 30 shows that 5 and 6 are factors of 30. Other whole number factors of 30 are 10 and 3, 2 and 15, 30 and 1, and 2, 3 and 5.

1. How many times is the factor '2' used in the product expression in A 18 above? ______ In A 16? ______

2. What factor is repeated more than once in example A 17? ______

3. What number is shown as a repeated factor in example A 13?

______

4. Write one or more product expressions for each of the following numerals. Show at least one repeated factor in each product expression.

Example 1: 16 = 4 x 4
16 = 2 x 2 x 2 x 2

Example 2: 24 = 2 x 2 x 6

The number of blanks will help you with some of them, although you may find several which do not fit the blanks.

a. 12 = ____ x ____ x ____
b. 18 = ____ x ____ x ____
c. 20 = ____ x ____ x ____
d. 28 = ____ x ____ x ____
e. 27 = ____ x ____ x ____
f. 50 = ____ x ____ x ____
g. 49 = ____ x ____
h. 25 = ____ x ____
i. 32 = ____ x ____ x ____
j. 36 = ____ x ____
k. 44 = ____ x ____ x ____
l. 1000 = ____ x ____ x ____, or
m. 72 =
n. 90 = ____ x ____ x ____
o. 60 = ____ x ____ x ____

p. 125 = ____ x ____ x ____

q. 100 =

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

D. There is a shorter way to write product expressions which show repeated factors. This way uses a numeral to tell the number of times a factor is repeated. Here is an example.

1. 4 x 4 x 4 is shortened to 4^3.

Notice that the symbol 4^3 is made up of two numerals. The large numeral at the left is called the base. It tells you what factor is being repeated. The small numeral at the upper right is called the exponent. It tells you how many times the repeated factor is used.

2. 7 x 7 is shortened to 7^2 because the factor 7 is used 2 times.
3. 2 x 2 x 2 x 2 is shortened to 2^4 because the factor 2 is used 4 times.
4. 5 x 5 x 5 is shortened to 5^3.
5. 3 x 3 x 3 x 3 is shortened to ("equals") 3^4.
6. 6 x 6 = 6^2

Fill in the blanks.

7. 8 x 8 = ____^{2}
8. 3 x 3 x 3 = 3-
9. 7 x 7 x 7 = ____- (Give the base and exponent)
10. 6 x 6 x 6 x 6 = ____ (Give the base and exponent)

These new symbols (4^3, 7^2, 2^4, etc.) are called exponent forms. 4^3 is read "four to the third power." 7^2 is read "seven to the second power." 2^4 is read "two to the fourth power."

In the expression

$$16 = 2 \times 2 \times 2 \times 2 = 2^4$$

16 is a decimal numeral, $2 \times 2 \times 2 \times 2$ is a product expression naming 16, while 2^4 is an exponent form of the same number.

Some numbers have more than one exponent form. For example:

11. 16 may be named by $2 \times 2 \times 2 \times 2 = 2^4$ or by
$4 \times 4 = 4^2$

12. 64 may be named by $8 \times 8 = 8^2$ or by
$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$ or
$\underline{\qquad} \times \underline{\qquad} \times \underline{\qquad} = \underline{\qquad}^{3}$

13. 81 may be named by $3 \times 3 \times 3 \times 3 = 3^4$ or by
$\underline{\qquad} \times \underline{\qquad} = \underline{\qquad}^{2}$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

E. Write a product expression and decimal numeral for each of the following.

Example: $5^3 = 5 \times 5 \times 5 = 125$

1. 5^2 2. 2^5
3. 20^2 4. 3^4
5. 6^3 6. 10^3

F. Write an exponent form for each of the following product expressions.

Example: $14 \times 14 = 14^2$

1. $9 \times 9 \times 9$
2. $6 \times 6 \times 6 \times 6$
3. 19×19
4. $23 \times 23 \times 23 \times 23 \times 23$
5. $11 \times 11 \times 11 \times 11$
6. $4 \times 4 \times 4 \times 4 \times 4 \times 4$

G. Write, for each of the following, a product expression containing one or more repeated factors. Also write a name in exponent form.

Example: $8 = 2 \times 2 \times 2 = 2^3$

1. 32 2. 81
3. 243 4. 144
5. 27 6. 216

H. Fill in the blanks in the following sentences.

1. The numeral 5^2 has exponent ____ and base ____.
2. The numeral ____ has exponent 4 and base 3.
3. The numeral 4^3 is the exponent form of the product expression

____ x ____ x ____.

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V. Multiplying Numbers Expressed in Exponent Form--meE

A. Sometimes it is necessary to name the product of two numbers expressed in exponent form. For example, $3^6 \times 3^3 = ?$

Whenever the base of each exponent form is the same, there is a rule which will help you find the answer quickly. (This rule does not apply when the base is different, such as in the problem $2^1 \times 4^2 = ?$).

Rule: Whenever two numbers expressed in exponent form have a common base, you may find their product by (1) using that same base for the base of the answer (also expressed in exponent form) and (2) adding the two exponents in the product expression to find the exponent in the answer.

Example 1: $3^6 \times 3^3 = 3^{(6 + 3)} = 3^9$

Example 2: $4^2 \times 4^4 = 4^6$

This rule works because the numbers expressed in exponent form are obtained by multiplying the base times itself as many times as the exponent indicates.

For example, in the expression $3^6 \times 3^3$, 3^6 really names the same number as $\underline{3 \times 3 \times 3 \times 3 \times 3 \times 3}$, while 3^3 names the same number as does $\underline{3 \times 3 \times 3}$. Therefore, $3^6 \times 3^3 =$ $\underline{(3 \times 3 \times 3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3)}$. Since the order of multiplication doesn't matter, parentheses are not needed where all the operations are multiplicative. They may be removed and we end up with $\underline{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}$, which by simple counting equals 3^9.

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B. Express the answer to each of the following product expressions in exponent form.

__Example__: $5^2 \times 5^5$

__Solution__: $5^2 \times 5^5 = (5 \times 5) \times (5 \times 5 \times 5 \times 5 \times 5)$

$= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$

$= 5^7$

$5^2 \times 5^5 = 5^7$

1. $6^2 \times 6^2$
2. $5^3 \times 5^2$
3. $7^1 \times 7^2$
4. $8^1 \times 8^1$
5. $2^8 \times 2^3$
6. $3^{11} \times 3^1$
7. $6^2 \times 6^4$
8. $5^2 \times 5^5$
9. $2^4 \times 2^8$
10. $2^{11} \times 2^4$
11. $10^1 \times 10^1$
12. $7^3 \times 7^3$
13. $7^2 \times 7^2$
14. $6^3 \times 6^4$
15. $2^4 \times 2^1$
16. $3^{10} \times 3^{10}$
17. $2^2 \times 2^5$
18. $4^2 \times 4^2$
19. $4^1 \times 4^3$
20. $5^3 \times 5^1$
21. $3^2 \times 3^4$
22. $2^3 \times 2^4$
23. $3^3 \times 3^3$
24. $3^2 \times 3^3$
25. $2^6 \times 2^1$
26. $5^2 \times 5^2$
27. $27^2 \times 27^3$
28. $243^2 \times 243^2$
29. $1728^2 \times 1728^4$
30. $\triangle^2 \times \triangle^3$

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C. Fill the blanks so that the following sentences are true.

1. $3^1 \times 3^4 = 3^{\_}$
2. $7^1 \times 7^1 = \_\_\_\_^2$
3. $6^2 \times 6^1 = \_\_\_\_^{\_}$
4. $2^4 \times \_\_\_\_^2 = 2^6$
5. $3^3 \times 3^{\_} = 3^6$
6. $5^{\_} \times 5^3 = 5^4$
7. $\_\_\_\_^2 \times 2^3 = 2^5$
8. $18^2 \times 18^3 = \_\_\_\_^5$
9. $21^3 \times 21^6 = \_\_\_\_$
10. $\square^3 \times \square^4 = \_\_\_\_$

D. Sort the following symbols into pairs which name the same number, using the rule that you have been taught.

Example: $12 + 4$ is a name for the number 16.

$9 - 3$ is also a name for the number 16.

1. $3^2 \times 3^3$
2. $2^5 \times 2^3$
3. $3^2 \times 3^{10}$
4. $3^2 \times 2^4$
5. $2^2 \times 6^2$
6. $3^4 \times 3^1$
7. $2^4 \times 2^4$
8. $2^2 \times 2^4$
9. $5^2 \times 5^2$
10. 5^4
11. $3^6 \times 3^6$
12. $2^5 \times 2^1$

E. Mark each of the following sentences true or false.

1. $2^4 \times 2^2 = 2^6$
2. $3^2 \times 3^3 = 3^5$
3. $2^3 \times 2^3 = 2^6$
4. $4^2 \times 4^3 = 4^4$
5. $10^4 \times 10^2 = 10^2$
6. $8^3 \times 8^2 = 8^5$
7. $4^6 \times 4^3 = 4^9$
8. $19^3 \times 19^5 = 19^8$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

F. Complete each of the following to make true sentences. Use only exponent forms.

1. $2^1 \times 2^4 =$ ____
2. $3^2 \times$ ____ $= 3^4$
3. $17^2 \times 17^4 =$ ____
4. ____ $\times 5^3 = 5^5$
5. $61^3 \times$ ____ $= 61^{10}$
6. $20^5 \times 20^3 =$ ____
7. $111^3 \times 111^4 =$ ____
8. ____ $\times 6^2 = 6^3$
9. $\square^5 \times \square^1 =$ ____
10. $\triangle^2 \times \triangle^5 =$ ____

G. The rule that you have been using (Rule: Whenever two numbers expressed in exponent form have a common base, you may find their product by (1) using that same base for the base of the answer (also expressed in exponent form) and (2) adding the two exponents in the product expression to find the exponent in the answer.) works only when you are multiplying numbers expressed in exponent forms which have the same base. It does not work in such problems as: $2^2 + 2^3$, $2^3 - 2^2$, $2^5 \div 2^2$ where addition, subtraction, and division are used, nor does it work in $4^3 \times 2^4$ where the base is different.

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

H. Mark each of the following mathematical sentences true or false. You may make any necessary computations in the space to the right.

1. $10^2 \times 10^2 = 10^4$
2. $4^4 \times 4^3 = 4^7$
3. $3^3 = 3 \times 3$
4. $3^3 = 3 \times 3 \times 3$
5. $2^3 \times 2^4 = 4^7$
6. $2^3 \times 2^4 = 2^{12}$
7. $11^4 \times 11^2 = 11^6$
8. $2^3 + 2^4 = 2^7$
9. $2^4 - 2^3 = 2^1$
10. $5^3 = 3 \times 5$
11. $2^3 \div 2^2 = 2^1$
12. $2^4 \div 2^2 = 2^2$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

VI. Dividing Numbers Expressed in Exponent Form--deE

A. As you might expect, there is a rule which will quickly help you find the answer to a problem such as $3^6 \div 3^3$. This rule is similar to the rule you have recently used in multiplying exponent forms in that it does not work when the base is different, such as in the problem $4^2 \div 2^1$. Rule: Whenever you divide two numbers expressed in exponent form, having a common base, you may find the answer by (1) using the same base for the base of the answer, and (2) subtracting the exponent of the divisor from the exponent of the dividend. (In the example $3^6 \div 3^3$, 3^3 is the divisor and 3^6 is the dividend. When written in this form, you can merely subtract the exponent of the second symbol from the exponent of the first.)

Example 1: $3^6 \div 3^3 = 3^{(6-3)} = 3^3$

Example 2: $4^5 \div 4^2 = 4^3$

B. Express the answer to each of the following expressions in exponent form.

Example: $5^7 \div 5^2 = 5^5$

1. $6^6 \div 6^3$
2. $2^9 \div 2^4$
3. $12^3 \div 12^1$
4. $3^{13} \div 3^{11}$
5. $5^6 \div 5^3$
6. $2^8 \div 2^6$
7. $4^7 \div 4^1$
8. $3^{10} \div 3^5$
9. $7^3 \div 7^2$
10. $8^5 \div 8^1$
11. $11^4 \div 11^2$
12. $10^6 \div 10^2$
13. $2^7 \div 2^2$
14. $4^4 \div 4^2$
15. $5^4 \div 5^2$
16. $3^7 \div 3^3$
17. $2^{10} \div 2^5$
18. $6^4 \div 6^1$

19. $10^4 \div 10^1$

20. $23^5 \div 23^2$

21. $104^4 \div 104^2$

22. $\square^6 \div \square^4$

C. Fill the blanks so that the following sentences are true.

1. $3^4 \div 3^1 = 3^{\_\_}$
2. $7^4 \div 7^2 = \_\_\_^2$
3. $6^5 \div 6^2 = \_\_\_^{\_\_}$
4. $2^6 \div 2^{\_\_} = 2^4$
5. $3^6 \div \_\_\_^3 = 3^3$
6. $5^{\_\_} \div 5^1 = 5^3$
7. $\_\_\_^5 \div 2^3 = 2^2$
8. $43^5 \times 43^3 = \_\_\_^8$
9. $13^6 \times 13^2 = \_\_\_$
10. $⬡^9 \div ⬡^1 = \_\_\_$

D. Mark each of the following sentences true or false.

1. $2^4 \div 2^2 = 2^2$
2. $7^6 \div 7^2 = 7^5$
3. $7^6 \div 7^2 = 7^4$
4. $4^9 \div 4^4 = 4^5$
5. $8^3 \div 8^1 = 8^2$
6. $10^6 \div 10^1 = 10^5$

E. Complete each of the following to make true sentences. Use only exponent forms.

1. $2^5 \div 2^2 = \_\_\_$
2. $12^4 \div 12^2 = \_\_\_$
3. $3^7 \div \_\_\_ = 3^3$
4. $\_\_\_ \div 19^2 = 19^7$
5. $43^6 \div 43^1 = \_\_\_$
6. $\triangle^8 \div \triangle^7 = \_\_\_$

(PLEASE DO NOT GO ON TO THE NEXT PAGE UNTIL TOLD TO DO SO.)

F. The rule which you have been using (Rule: Whenever you divide two numbers expressed in exponent form, having a common base, you may find the answer by (1) using the same base for the base of the answer, and (2) subtracting the exponent of the divisor from the exponent of the dividend.) works only when the numbers you are using in the division process are expressed in exponent forms which have the same base. It does not work in such problems as $3^3 + 3^2$, $3^3 - 3^2$, or $3^5 \times 3^2$ where addition,subtraction, or multiplication processes are used, nor does it work with such problems as $4^3 \div 2^2$, where the base is different.

G. Mark each of the following mathematical sentences true or false. You may make any necessary computations in the space to the right.

1. $9^4 \div 9^2 = 9^2$
2. $3^4 \div 3^2 = 0^2$
3. $4^6 \div 4^3 = 4^2$
4. $5^4 \div 5^2 = 5^2$
5. $6^3 \div 2^2 = 3^1$
6. $7^{11} \div 7^1 = 7^1$
7. $8^8 \div 8^5 = 8^3$
8. $10^{20} \div 10^5 = 10^{15}$

APPENDIX III

Instruments

PAGE AND ABBREVIATION KEY FOR EACH INSTRUMENT CONTAINED IN APPENDIX III

dpCT	-	Distributive principle concept test	A3.04
eCT	-	Exponent concept test	A3.06
raCT	-	Real number addition concept test	A3.08
rmCT	-	Real number multiplication concept test	A3.11
TOT-CT	-	Total concept test	A3.03
dpRT	-	Distributive principle retention test	A3.41
eRT	-	Exponent retention test	A3.43
raRT	-	Real number addition retention test	A3.45
rmRT	-	Real number multiplication retention test	A3.48
TOT-RT	-	Total retention test	A3.40
cbAT	-	Choice behavior attitude test	A3.14
sdAT	-	Semantic differential attitude test	A3.23
asAT	-	Attitude statement attitude test	A3.28
TOT-AT*	-	Total attitude test	
cTT	-	Concept transfer test	A3.51
cnTT	-	Concept negative transfer test	A3.56
nsTT	-	Number series transfer test	A3.57
apTT	-	Associative principle transfer test	A3.62
stR	-	Student rating of teaching behavior	A3.35
olR	-	Observer live rating of teaching behavior	A3.66
taR	-	Tape rating of teaching behavior	A3.66

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*Not used in this Appendix, but used elsewhere.

Method of Administration - dpCT, eCT, raCT, and rmCT

The examiner first hands out dpCT and then reads the instructions aloud while the students read them silently. Stress should be placed upon the use of the space at the right for all computations, however trivial. The examiner then answers any procedural questions, taking care to give no hint on how to work the problems.

No time limit is imposed. This is an unspeeded power test. When all class members have finished, the examiner collects the booklets and hands out eCT. The same procedure is followed, in order, with raCT and rmCT.

Scoring - dpCT, eCT, raCT, and rmCT

dpCT	- one point for each correct answer, with an additional four points if they solve it by use of the dpma (as evidenced by the computational scratchwork	25
eCT	- two points for each correct answer	20
raCT	- one point for each correct answer	15
rmCT	- one point for each correct answer	15
	TOTAL	75

A3.04

Name ______________________________

Mathematics Concept Test - dp CT

This is a test to see how many of the following problems you already know how to do. You are not expected to be able to solve all of the problems. Merely do your best and try to work every problem that you have an idea how to work. The teacher will tell you when to stop.

Please show all of your computation and scratchwork (even that which you normally do in your head). Put your work in the space at the right of the page. Then put the answer on the line shown in the problem itself. Do not erase any of your work.

Example:

2 x 14 + 8 x 14 = 140

$$\begin{array}{r} 14 \\ \times 2 \\ \hline 28 \end{array} \qquad \begin{array}{r} {}^{3} \\ 14 \\ \times 8 \\ \hline 112 \end{array} \qquad \begin{array}{r} {}^{1} \\ 112 \\ +28 \\ \hline 140 \end{array}$$

When you have finished working, turn your paper over.

Start Here:

1. 14 x 88 + 14 x 12 = _______

2. 6 x 29 + 4 x 29 = _______

3. $283 \times \frac{1}{4} + 283 \times \frac{3}{4} =$ _______

4. 3 x 16 + 3 x 14 = _______

5. 18 x 3 + 15 x 3 = _______

A3.06

Name ______________________________

Mathematics Concept Test - eCT

This is a test to see how many of the following problems you already know how to do. You are not expected to be able to solve all of the problems. Merely do your best and try to work every problem that you have an idea how to work. The teacher will tell you when to stop.

Please show all of your computation and scratchwork in the space at the right of the page. Then put the answer on the line shown in the problem itself. Do not erase any of your work.

Example:

180 ÷ 10 = 18

$$\begin{array}{r} 18 \\ 10\overline{)180} \\ \underline{10} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

When you have finished working, turn your paper over.

Start Here:

1. $2^3 =$ ______

2. $8^2 =$ ______

3. $1^5 =$ ______

Put the rest of the answers in exponent form

4. $3^2 \times 3^3 =$ ______

5. $2^4 \div 2^2 =$ ______

6. $5^2 \times 5^1 =$ ______

7. $9^3 \div 9^1 =$ ______

8. $2^5 \times 2^2 =$ ______

9. $5^4 \div 5^2 =$ ______

10. $6^2 \times 2^2 =$ ______

Name ______________________

Mathematics Concept Test - ra CT

This is a test to see how many of the following problems you already know how to do. You are not expected to be able to solve all of the problems. Merely do your best and try to work every problem that you have an idea how to work. The teacher will tell you when to stop.

Please show all of your computation and scratchwork in the space at the right of the page. Then put the answer on the line shown in the problem itself. Do not erase any of your work.

Example:

8 + 14 = 22

$$\begin{array}{r} \overset{1}{1}4 \\ +\ 8 \\ \hline 22 \end{array}$$

When you have finished working, turn your paper over.

A3.09

Start here:

1. ${}^{+}9 + {}^{-}10 =$ ______

2. ${}^{+}3 + {}^{+}8 =$ ______

3. ${}^{-}3 + {}^{-}2 =$ ______

4. ${}^{-}18 + {}^{+}17 =$ ______

5. ${}^{+}5 + {}^{-}5 =$ ______

6. ${}^{+}11 +$ ______ $= 0$

7. $0 + {}^{+}10 =$ ______

8. ${}^{-}3 +$ ______ $= {}^{+}3$

9. $^{-}4 +$ ________ $= {}^{+}8$

10. ________ $+ {}^{-}2 = {}^{-}4$

11. $^{+}12 +$ ________ $= {}^{+}22$

12. $({}^{+}9 + {}^{-}9) + 0 =$ ________

13. $^{+}7 + ({}^{-}3 + {}^{-}8) =$ ________

14. $({}^{+}9 + {}^{-}5) + {}^{-}24 =$ ________

15. $({}^{-}8 + {}^{+}3) + {}^{-}12 =$ ________

Name ______________________________

Mathematics Concept Test - rm CT

This is a test to see how many of the following problems you already know how to do. You are not expected to be able to solve all of the problems. Merely do your best and try to work every problem that you have an idea how to work. The teacher will tell you when to stop.

Please show all of your computation and scratchwork in the space at the right of the page. Then put the answer on the line shown in the problem itself. Do not erase any of your work.

Example:

8 x 14 = 112

$$\begin{array}{r} {}^{3} \\ 14 \\ \underline{\times\ 8} \\ 112 \end{array}$$

When you have finished working, turn your paper over.

A3.12

Start Here:

1. $^{+}4 \times {}^{-}2 =$ ________

2. $^{-}4 \times {}^{-}2 =$ ________

3. $^{+}4 \times {}^{+}2 =$ ________

4. $^{-}4 \times {}^{+}2 =$ ________

5. $^{+}7 \times {}^{+}3 =$ ________

6. $^{-}7 \times {}^{-}3 =$ ________

7. $^{+}7 \times {}^{-}3 =$ ________

8. $^{-}7 \times {}^{+}3 =$ ________

9. $^{-}8 \times 0 = \underline{\qquad\qquad}$

10. $^{+}5 \times \underline{\qquad\qquad} = {}^{-}20$

11. $\underline{\qquad\qquad} \times {}^{+}7 = {}^{+}21$

12. $^{-}7 \times \underline{\qquad\qquad} = {}^{-}28$

13. $^{+}4 \times ({}^{-}6 \times {}^{+}2) = \underline{\qquad\qquad}$

14. $({}^{+}5 \times {}^{-}3) \times {}^{-}3 = \underline{\qquad\qquad}$

15. $^{-}1 \times ({}^{-}1 \times {}^{-}1) = \underline{\qquad\qquad}$

Method of Administration - cbAT

(The examiner first hands out the cbAT booklet, instructing the students to fill in their name, teacher, and school on the first page. The examiner then reads aloud the following instructions.)

This is to find out how boys and girls feel about what they study in school. It is necessary that you fill in as many of the blanks on each page as you can in the time allowed.

Please turn to the second page. Notice that this page is divided in half. Read all the questions on both sides. Raise your hand when you have finished reading the questions. (Examiner waits until all hands are up.) Now decide which side of the page you would rather do. Cross out the whole side you do not want to do with a large X-mark. Then fill in all of the blanks on the other side. Do not turn the page until told to do so. You may begin.

(The examiner observes to see that pupils are following instructions. When about 80 per cent of the pupils are finished, the examiner continues.)

Time: Now go on to the next page and do it in the same way. First read all of the questions on both sides of the page. Then cross out the side you do not want to do and fill in the blanks on the side you chose to do. You may begin.

(The examiner should judge when approximately 80 per cent of the pupils are finished and then call time.

Pupils are not permitted to change their minds once they have selected which side of the page they are to work on. If a student says he can't do a problem or expresses a desire to change sides, the examiner says, in effect, "Do the best you can with the side you have chosen." When the first four pages of the test have been completed, the examiner says:)

Now turn to the final page. We do not know at this time whether or not your choice can be given to you, but if you are given the choice whether to have more arithmetic or less arithmetic for the remainder of the school year, which will you choose, more or less? If you choose "more," put a check mark on the line above the word "more." If you choose "less," put a mark above "less." Now hand your papers in.

cbAT

Name ______________________________

Teacher ______________________________

School ______________________________

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Fill the frames correctly.	Fill the blanks correctly.
$\begin{array}{r} 34\square 2 \\ +\ \square 967 \\ \hline 7\square 8\square \end{array}$	The opposite of float is ______________.
$\begin{array}{r} 10000 \\ -\ 1\square 37 \\ \hline \square 4\square 3 \end{array}$	The inventor of the electric light was ________________.
$\begin{array}{r} 2937 \\ \times\ \ 8 \\ \hline 23\square 9\square \end{array}$	The upper stream of a river flows ________ than the lower stream.
$\begin{array}{r} \square 1 \square \\ 7\overline{)833} \end{array}$	I can speak with you some other time more leisurely, since I am ______________ today.

STOP (do not turn the page until told to do so).

How is iron like silver?

A boy cut off a piece of rope 4 ft. long from a 20 ft. rope. How many pieces of 2 ft. rope can be cut from the rest of the rope?

Make a sentence with these three words: work, money man.

How many hours will it take to drive a team 66 miles at the rate of 6 miles an hour?

STOP (do not turn the page until told to do so).

John bought 4 apples at 12 cents each, 3 oranges at 10 cents each. How much change should John get from a dollar bill:

John had $4.50. He bought one book at $1.75 and one pencil box at 75 cents and one dozen pencils at 4 cents each. How much did he have left?

Fill the blanks correctly.

What is the opposite of push? ______________

A star is ____________ than the moon.

The discoverer of Radium was ________________

________________ tomorrow is Sunday, we shall all go hiking.

STOP (do not turn the page until told to do so).

Make a sentence with these three words: children, ball, river.

How is a ship like an automobile?

Fill the frames correctly.

$$\begin{array}{r} 2\,5\,\square\,6 \\ +\ \square\,8\,2\,3 \\ \hline 7\,\square\,6\,\square \end{array}$$

$$\begin{array}{r} 7\,2\,8\,6 \\ -\ 4\,\square\,5\,\square \\ \hline 2\,3\,3\,3 \end{array}$$

$$\begin{array}{r} 6\,3\,4\,8 \\ \times\ \ \ \ \ 7 \\ \hline \square\,4\,\square\,3\,\square \end{array}$$

$$\begin{array}{r} \square\,6\,\square \\ 6\,\overline{)\,9\,8\,4} \end{array}$$

STOP (do not turn the page until told to do so).

If you are given the choice whether to have more arithmetic or less arithmetic for the remainder of the school year, which will you choose?

________ **more**   ________ **less**

Scoring - cbAT

No effort was made to score the correctness of the responses to either the verbal or arithmetic problems. Each page was scored on the basis of which side of the page the children chose to work on. If the "arithmetic" side was chosen, the page was scored $^{+}5$. If the "verbal" side was chosen, the page was scored $^{-}5$. This scoring system was used on pages A3.17, A3.18, A3.19, and A3.20.

On page A3.21, $^{+}10$ was assigned to a response of "more" and $^{-}10$ was assigned to a response of "less."

The summation of scores on pages A3.17, A3.18, A3.19, and A3.20 gives a possible range of $^{+}20$ to $^{-}20$ with a set interval of 10. Addition of page A3.21 increases the possible range to $^{+}30$ to $^{-}30$ with the interval remaining constant.

Method of Administration - sdAT

(The examiner reads aloud the following instructions, using the blackboard to demonstrate the examples.)

We want to find out how pupils feel about things that have to do with their arithmetic classes. In order to do this, we will ask you to tell us how you feel about arithmetic.

Here is how you do it. Suppose we want to find out how you feel about "numbers." We would then write "numbers" at the top of the page. (Write it.) Then there will be pairs of words like this (write "bitter" and "sweet" at ends of scale) with five spaces in between. If you think that "numbers" are very much like "bitter" you should put a mark on that end (demonstrate).

bitter __x__ | _____ | _____ | _____ | _____ sweet

or if you think "numbers" are very much like "sweet" you put it at the other end. (Demonstrate, first erasing previous mark.)

bitter _____ | _____ | _____ | _____ | __x__ sweet

If you feel that "numbers" are closely, but not exactly, related to the word at one end of the line you would make a mark a little nearer the center of the line. Like this, if it's a little like "bitter," (demonstrate).

bitter _____ | __x__ | _____ | _____ | _____ sweet or like this if it's a little like "sweet." (demonstrate)

bitter ____ | ____ | ____ | __x__ | ____ sweet

If you feel completely neutral about a word, or if the words on the ends of the line do not seem to be related at all to the other word, then you should put your check mark in the middle space. (Demonstrate)

bitter ____ | ____ | __x__ | ____ | ____ sweet

The direction in which you check depends on which end of the line seems more like the word you are judging, and the distance from the word at the end depends on how closely you think it is like that word.

Be sure that you put your check mark in the spaces and not on the lines. (demonstrate)

This   Not this

bitter ____ | __X__ | ____ X ____ | ____ sweet

Be sure that you have a check mark on every line for every page. Do not omit any. And be sure that there is only one check mark on each line.

Sometimes you may feel as though you've had the same words before. This will not be the case, so do not look back and forth through the items. Do not try to remember how you checked similar items earlier. Work at fairly high speed. Do not worry or puzzle over the items. Give your first impressions, the immediate "feelings" about each one. On the other hand, please do not be careless because we want your true impression.

Do you have any questions? (The examiner answers any questions about how to use the scale but gives no information on how to answer any item. When all questions are answered, the examiner hands out sheet sdAT and tells the students to begin immediately. No time limit should be imposed. Allow everyone to complete the instrument.)

sdAT Name ______________________________ A3.26

Arithmetic

fast	____	____	____	____	____	slow
sad	____	____	____	____	____	happy
nice	____	____	____	____	____	awful
small	____	____	____	____	____	large
unpleasant	____	____	____	____	____	pleasant
clear	____	____	____	____	____	hazy
weak	____	____	____	____	____	strong
interesting	____	____	____	____	____	boring
unfair	____	____	____	____	____	fair
clean	____	____	____	____	____	dirty
sharp	____	____	____	____	____	dull
important	____	____	____	____	____	unimportant
sour	____	____	____	____	____	sweet
cold	____	____	____	____	____	hot
good	____	____	____	____	____	bad
worthless	____	____	____	____	____	valuable
meaningful	____	____	____	____	____	meaningless
long	____	____	____	____	____	short
distasteful	____	____	____	____	____	tasty

Scoring - sdAT

Six of the nineteen scales were not used in the scoring. These included fast - slow, small - large, weak - strong, sharp - dull, cold - hot, and long - short.

The remaining thirteen items were scored as follows.

sad	-2	-1	0	+1	+2	happy
nice	+2	+1	0	-1	-2	awful
unpleasant	-2	-1	0	+1	+2	pleasant
clear	+2	+1	0	-1	-2	hazy
interesting	+2	+1	0	-1	-2	boring
unfair	-2	-1	0	+1	+2	fair
clean	+2	+1	0	-1	-2	dirty
important	+2	+1	0	-1	-2	unimportant
sour	-2	-1	0	+1	+2	sweet
good	+2	+1	0	-1	-2	bad
worthless	-2	-1	0	+1	+2	valuable
meaningful	+2	+1	0	-1	-2	meaningless
distasteful	-2	-1	0	+1	+2	tasty

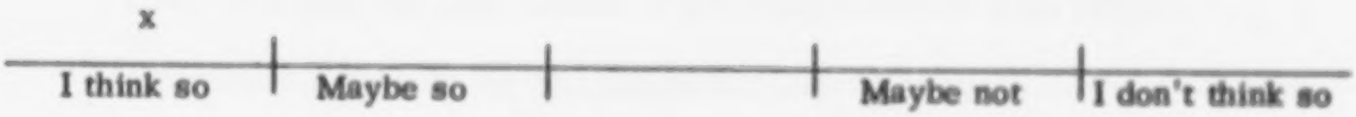
possible range - +26 to -26

Method of Administration - asAT

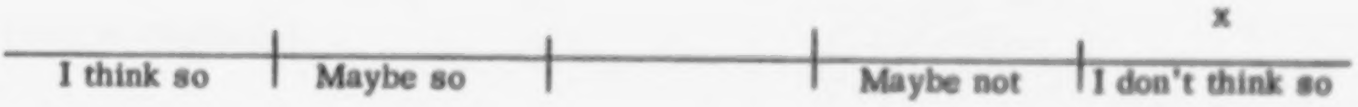
(The examiner first hands out copies of asAT and asks the students to put their names at the top. The examiner then reads aloud the following instructions, using the blackboard to demonstrate how to mark.)

On this page are several long lines. Notice that one end of each line is labeled "I think so" and the other is "I don't think so." I am going to read you several statements. For every statement I read you should put a check mark on any part of the line which tells how you feel about that statement.

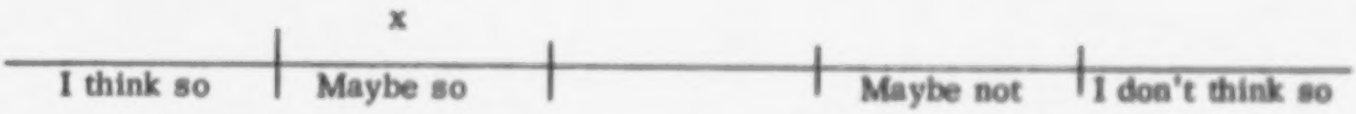
Here is how you do it. Suppose I read the statement, "School is fun." If you think that school is fun, you should put a mark on the part of the line above the words "I think so;" (demonstrate)



or if you think school is not fun, you put it at the other end. (demonstrate)



If you feel that school may be fun, but you aren't quite certain, put a mark on the part of the line above the words, "maybe so;" (demonstrate)



or if you feel school may not be fun, but you aren't quite certain, put a mark above the words, "maybe not." (demonstrate)

			x	
I think so	Maybe so		Maybe not	I don't think so

If you feel completely neutral about the statement and cannot decide whether or not school is fun, put a mark above the blank center space. (demonstrate)

		x		
I think so	Maybe so		Maybe not	I don't think so

Are there any questions? (The examiner answers any questions about how to use the scale. When all questions are answered, the examiner continues.)

I will now read the statements. I will repeat each of them twice.

(The examiner reads the 32 statements, setting his pace so the class can complete each item. The examiner reads the number of the statement and the statement itself, but omits "+" or "-" which precedes each statement. No time limit is imposed.)

1. + I feel especially capable when doing arithmetic problems
2. + It is easy to get good marks in arithmetic.
3. - Arithmetic problems often scare me.
4. - Arithmetic requires too much of one's time.
5. + Arithmetic is very useful in life.
6. - Arithmetic often makes me feel irritable and angry.
7. - What we are asked to do in arithmetic is just too much.
8. + I feel relaxed when doing arithmetic problems.
9. - Arithmetic often makes my mind go blank.
10. - Time drags in an arithmetic lesson.
11. + I feel sure of myself when doing arithmetic problems.
12. + More school time should be given to arithmetic.
13. - Arithmetic is only a drill subject.
14. + I usually feel calm when doing my arithmetic problems.
15. + Arithmetic requires less time and effort than other subjects.
16. - I often feel sad when doing arithmetic problems.
17. + I think my mind works well when doing arithmetic problems.
18. + Arithmetic problems are a good way to spend one's spare time.

19. - I sometimes feel like running away from my arithmetic problems.

20. - Too much stress is put on the correct answer in arithmetic.

21. + Time goes fast in an arithmetic lesson.

22. - I hate numbers and wish they had never been invented.

23. - It is difficult to succeed in arithmetic.

24. + I usually feel happy when doing arithmetic problems.

25. - Arithmetic problems often make me feel lower than other people.

26. - A person has little use for arithmetic.

27. + I like to struggle with an arithmetic problem.

28. + Arithmetic is a necessary tool for higher studies.

29. - In school there should be less arithmetic and more time for other subjects.

30. + I like anything with numbers in it.

31. + To be exact in arithmetic is good training for one's future work.

32. - Arithmetic problems often make me feel upset.

asAT Name ______________________________ A3.32

1.	I think so	Maybe so		Maybe not	I don't think so
2.	I think so	Maybe so		Maybe not	I don't think so
3.	I think so	Maybe so		Maybe not	I don't think so
4.	I think so	Maybe so		Maybe not	I don't think so
5.	I think so	Maybe so		Maybe not	I don't think so
6.	I think so	Maybe so		Maybe not	I don't think so
7.	I think so	Maybe so		Maybe not	I don't think so
8.	I think so	Maybe so		Maybe not	I don't think so
9.	I think so	Maybe so		Maybe not	I don't think so
10.	I think so	Maybe so		Maybe not	I don't think so
11.	I think so	Maybe so		Maybe not	I don't think so
12.	I think so	Maybe so		Maybe not	I don't think so
13.	I think so	Maybe so		Maybe not	I don't think so
14.	I think so	Maybe so		Maybe not	I don't think so
15.	I think so	Maybe so		Maybe not	I don't think so
16.	I think so	Maybe so		Maybe not	I don't think so

17.	I think so	Maybe so		Maybe not	I don't think so
18.	I think so	Maybe so		Maybe not	I don't think so
19.	I think so	Maybe so		Maybe not	I don't think so
20.	I think so	Maybe so		Maybe not	I don't think so
21.	I think so	Maybe so		Maybe not	I don't think so
22.	I think so	Maybe so		Maybe not	I don't think so
23.	I think so	Maybe so		Maybe not	I don't think so
24.	I think so	Maybe so		Maybe not	I don't think so
25.	I think so	Maybe so		Maybe not	I don't think so
26.	I think so	Maybe so		Maybe not	I don't think so
27.	I think so	Maybe so		Maybe not	I don't think so
28.	I think so	Maybe so		Maybe not	I don't think so
29.	I think so	Maybe so		Maybe not	I don't think so
30.	I think so	Maybe so		Maybe not	I don't think so
31.	I think so	Maybe so		Maybe not	I don't think so
32.	I think so	Maybe so		Maybe not	I don't think so

Scoring - asAT

The 32 statements which were read orally to the students included both positive and negative statements. On positive statements, the following scoring system was used.

$^{+}2$	$^{+}1$	0	$^{-}1$	$^{-}2$
I think so	Maybe so		Maybe not	I don't think so

On negative statements, the following scoring system was used.

$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$
I think so	Maybe so		Maybe not	I don't think so

Below is a listing, by number, of the positive and negative statements.

1. +
2. +
3. -
4. -
5. +
6. -
7. -
8. +
9. -
10. -
11. +
12. +
13. -
14. +
15. +
16. -
17. +
18. +
19. -
20. -
21. +
22. -
23. -
24. +
25. -
26. -
27. +
28. +
29. -
30. +
31. +
32. -

Possible range - $^{+}64$ to $^{-}64$.

Method of Administration - stR

(The examiner hands out the stR booklets and then reads aloud the following instructions.)

We wish to know the kinds of things teachers do in teaching arithmetic. In this inventory, each of you will tell how often your teacher does certain things. As soon as all of you have completed your answers, your sheets will be collected, placed in a sealed envelope, and sent directly to the "Scoring Center." After your sheets have been scored, they will be destroyed. No one will ever know what answers you gave since your name will be taken off the sheet.

Now read the directions on your sheet silently while I read them aloud.

(The examiner reads aloud the directions and checks to make certain all students have responded correctly to sample problem No. 2. The examiner then asks for and answers any questions about how to use the marking system. When all questions are answered, the examiner continues.)

Now you are ready to begin the next page. Altogether there are twenty-five statements. Read one at a time and mark on your answer sheet your decision about how often your teacher does it. Sometimes you may feel as though you've had the same item before on the test. This will not be the case, so do not look back and forth through the items. Do not try to remember how you checked similar items earlier in the test. Work quickly, but carefully. You may begin.

(The examiner helps students read statements which they cannot read. Otherwise no help is given. No time limit is imposed. When all students have finished, the examiner collects the papers.)

stR Name ______________________________

Directions:

Read each sentence carefully and decide how often it is done in your class. Mark as follows for each statement you read:

Mark "A" if your teacher almost never does it in your class.

Mark "B" if your teacher seldom does it (less than half of the time).

Mark "C" if your teacher sometimes does it (about half of the time).

Mark "D" if your teacher often does it (more than half the time).

Mark "E" if your teacher almost always does it in your class.

Here is an example. Suppose the statement is: Our teacher ____________ plays with us at recess time.

Suppose you decided the best description of how often your teacher plays with you at recess time is "often." Then you would place a "D" in the blank. Like this:

1. Our teacher __D__ plays with us at recess time.

Now suppose you decided the best description of how often your teacher plays with you at recess time is "sometimes." What letter would you place in the blank?

2. Our teacher ________ plays with us at recess time.

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO

A - Almost Never
B - Seldom
C - Sometimes
D - Often
E - Almost Always

---

1. When an answer is wrong, our teacher ________ tells us immediately.
2. Our teacher ________ tells us how to do a problem.
3. When we ask the teacher how to work a problem, he/she ________ tells us.
4. Our teacher ________ lets us talk to each other about our arithmetic.
5. When a rule won't apply to all problems, our teacher ________ warns us when not to use it.
6. Our teacher ________ acts surprised when a problem doesn't work out easily.
7. Our teacher ________asks us to use the rules he/she has taught us.
8. When one of us explains how he got his answer to another class member, our teacher is ________ unhappy.
9. Our teacher ________ gives us the answer to everything he is asked.
10. In our class, we are ________ given trick questions where a rule won't work.
11. Our teacher ________ acts in doubt as to how to work a problem.
12. When someone works a problem quickly, our teacher ________ acts surprised.
13. Our teacher ________ gives us a rule to use for solving new kinds of problems.
14. Our teacher ________ asks us not to tell others how to do the problem.

A - Almost Never
B - Seldom
C - Sometimes
D - Often
E - Almost Always

---

15. In our class, we are _______ given a direct answer to our questions.

16. Our teacher _______ repeats rules for solving problems.

17. Our teacher _______ gives the right answer.

18. When a new problem comes up, our teacher _______ shows us exactly how to do it.

19. Our teacher _______ lets us try for ourselves to use a rule on a problem even when he/she knows it does not work.

20. Our teacher _______ corrects a wrong answer immediately.

21. In our class, we are _______ permitted to consult with our classmates.

22. When a student discovers a wrong answer, our teacher _______ acts surprised.

23. Our teacher is _______ happy when one of us tells another how to do a problem.

24. When we ask the teacher how to work a problem, he/she _______ lets us try to work it out for ourselves.

25. Our teacher _______ gives us a chance to catch our own errors before he/she points them out.

Scoring - stR

The students' responses (almost never, seldom, sometimes, often, or almost always) were counted for each item and the percentages of the students answering the items each way were determined. On the "low discovery-high expository" items the percentages were weighted by assigning 4 for almost never, 3 for seldom, 2 for sometimes, 1 for often, and 0 for almost always. On the "high discovery-low expository" items this scale was reversed with a weighting of 0 for almost never, 1 for seldom, 2 for sometimes, 3 for often, and 4 for almost always. The weighted percentages for the item were added and divided by 4. The quotient became the index, with a possible range of 0 for the "low discovery-high expository" end of the continuum to 100 for the "high discovery-low expository" end of the continuum.

Suppose the item were number 1:

1. When an answer is wrong, our teacher ______ tells us immediately.

Suppose the item were responded to by 25 students as follows:

	Almost never	Seldom	Sometimes	Often	Almost Always
Item 1	0	1	4	10	10
%	0	4	16	40	40
Weight	4	3	2	1	0
Product (% x Wt)	0	12	32	40	0
Total			84		
Index (divide by 4)			21		

Below is a listing, by number, of the "high discovery-low expository" (hd) items and the "low discovery-high expository" (ld) items.

1. ld	6. hd	11. hd	16. ld	21. ld
2. ld	7. ld	12. hd	17. ld	22. hd
3. ld	8. hd	13. ld	18. ld	23. ld
4. ld	9. ld	14. hd	19. hd	24. hd
5. ld	10. hd	15. ld	20. ld	25. hd

Method of Administration - dpRT, eRT, raRT, and rmRT

The procedures and instructions used in administering these tests are identical to those used in administering dpCT, eCT, raCT, and rmCT. These are shown on page A3.03.

Scoring - dpRT, eRT, raRT, and rmRT

Scoring of these tests is identical to the scoring of the comparable concept tests shown on page A3.03.

Name ______________________________

Mathematics Concept Test - op[illegible]

This is a test to see how many of the following problems you already know how to do. You are not expected to be able to solve all of the problems. Merely do your best and try to work every problem that you have an idea how to work. The teacher will tell you when to stop.

Please show all of your computation and scratchwork (even that which you normally do in your head). Put your work in the space at the right of the page. Then put the answer on the line shown in the problem itself. Do not erase any of your work.

Example:

2 x 14 + 8 x 14 = 140

$$\begin{array}{r} 14 \\ \times 2 \\ \hline 28 \end{array} \qquad \begin{array}{r} {}^{3}14 \\ \times 8 \\ \hline 112 \end{array} \qquad \begin{array}{r} \overset{1}{1}12 \\ +28 \\ \hline 140 \end{array}$$

When you have finished working, turn your paper over.

Start Here:

1. $17 \times 81 + 17 \times 19 =$ ______

2. $3 \times 37 + 7 \times 37 =$ ______

3. $189 \times \frac{1}{5} + 189 \times \frac{4}{5} =$ ______

4. $3 \times 16 + 3 \times 17 =$ ______

5. $36 \times 3 + 24 \times 3 =$ ______

Name ______________________________

Mathematics Concept Test - eRT

This is a test to see how many of the following problems you already know how to do. You are not expected to be able to solve all of the problems. Merely do your best and try to work every problem that you have an idea how to work. The teacher will tell you when to stop.

Please show all of your computation and scratchwork in the space at the right of the page. Then put the answer on the line shown in the problem itself. Do not erase any of your work.

Example:

$180 \div 10 =$ 18

```
    18
10)180
   10
    80
    80
     0
```

When you have finished working, turn your paper over.

Start Here:

1. $2^4 =$ _____

2. $6^2 =$ _____

3. $1^4 =$ _____

Put in the rest of the answers in exponent form

4. $3^2 \times 3^2 =$ _____

5. $2^5 \div 2^2 =$ _____

6. $2^4 \times 2^3 =$ _____

7. $8^3 \div 8^1 =$ _____

8. $7^2 \times 7^1 =$ _____

9. $5^3 \div 5^2 =$ _____

10. $4^2 \times 2^2 =$ _____

Name ______________________________

Mathematics Concept Test - raRT

This is a test to see how many of the following problems you already know how to do. You are not expected to be able to solve all of the problems. Merely do your best and try to work every problem that you have an idea how to work. The teacher will tell you when to stop.

Please show all of your computation and scratchwork in the space at the right of the page. Then put the answer on the line shown in the problem itself. Do not erase any of your work.

Example:

8 + 14 = 22

$$\begin{array}{r} {}^{1}14 \\ +\ 8 \\ \hline 22 \end{array}$$

When you have finished working, turn your paper over.

Start here:

1. $^{-}15 + {}^{+}14 =$ ______

2. $^{-}5 + {}^{-}4 =$ ______

3. $^{+}9 + {}^{+}4 =$ ______

4. $^{+}10 + {}^{-}9 =$ ______

5. $^{+}6 + {}^{-}6 =$ ______

6. $^{+}12 +$ ______ $= 0$

7. $0 + {}^{+}9 =$ ______

8. $^{-}5 +$ ______ $= {}^{+}5$

9. $^-2 + \underline{\qquad} = {}^+4$

10. $\underline{\qquad} + {}^-4 = {}^-8$

11. $^+10 + \underline{\qquad} = {}^+22$

12. $(^-11 + {}^+11) + 0 = \underline{\qquad}$

13. $^+5 + (^-6 + {}^-7) = \underline{\qquad}$

14. $(^-5 + {}^+8) + {}^-23 = \underline{\qquad}$

15. $(^+2 + {}^-9) + {}^-11 = \underline{\qquad}$

Name ____________________

Mathematics Concept Test - rmRT

This is a test to see how many of the following problems you already know how to do. You are not expected to be able to solve all of the problems. Merely do your best and try to work every problem that you have an idea how to work. The teacher will tell you when to stop.

Please show all of your computation and scratchwork in the space at the right of the page. Then put the answer on the line shown in the problem itself. Do not erase any of your work.

Example:

8 x 14 = 112

$$\begin{array}{r} {}^{3}14 \\ \times\ 8 \\ \hline 112 \end{array}$$

When you have finished working, turn your paper over.

Start Here:

1. ${}^{-}2 \times {}^{+}5 =$ ______

2. ${}^{+}2 \times {}^{+}5 =$ ______

3. ${}^{+}2 \times {}^{-}5 =$ ______

4. ${}^{-}2 \times {}^{-}2 =$ ______

5. ${}^{+}6 \times {}^{+}3 =$ ______

6. ${}^{-}6 \times {}^{+}3 =$ ______

7. ${}^{-}6 \times {}^{-}3 =$ ______

8. ${}^{+}6 \times {}^{-}3 =$ ______

PRECEDING PAGE MISSING

Method of Administration - cTT

(The examiner reads aloud the following instructions:)

This is a test to see how well you are able to work certain kinds of problems. It is important that you try to work all of the problems in the easiest possible way. We would also like you to show all of your computation on the test paper. I am now going to pass out the tests. Please leave them facedown on your desks.

(The examiner then passes out the cTT booklets facedown on the desks. When all booklets have been handed out the examiner says:)

Turn your booklets over and put your name at the top of the first page. Read the directions on page A3.52 silently while I read them aloud. (Examiner reads the directions aloud.) These directions apply to problems 1 through 12. Are there any questions? (Examiner answers noncomputational questions, taking care to give no information on how the problems are to be approached.) Now read the directions in the center of page A3.53 silently while I read them aloud. (Examiner reads directions aloud.) These directions apply to problems 13 through 22.

(The examiner then follows the same procedure with the directions on pages A3.54 and A3.55. When all questions have been answered, the pupils are allowed to begin. No time limit is imposed. This is an unspeeded power test.)

Simplify the following expressions in the most simple manner possible. Show all of your computation in the space to the right, even that which you normally do in your head.

---

1. $(6 \times 9) + 16 =$ ______

2. ${}^{-}2 \times {}^{-}12 + {}^{+}2 \times {}^{-}12 =$ ______

3. ${}^{+}8 \times {}^{-}37 + {}^{+}2 \times {}^{-}37 =$ ______

4. $87 \times 13 + 12 \times 13 =$ ______

5. $132 + 13 =$ ______

6. $6 \times 47 + 1 \times 47 + 3 \times 47 =$ ______

7. $9 \times 38 + 70 \times 38 + 21 \times 38 =$ ______

8. ${}^{+}69 \times {}^{-}4 + {}^{+}69 \times {}^{+}14 =$ ______

9. $(^-31 + {}^+31) + (^-4 \times 0) =$ ______

10. $99 \times \frac{1}{2} + 1 \times \frac{1}{2} =$ ______

11. $(4 + 6) - (8 - 2) =$ ______

12. $27 \times 5 + 27 \times 4 + 27 \times 1 =$ ______

Mark the following expressions true or false. Show any scratchwork in the space to the right. Remember that " > " means "is greater than and "< " means "is less than."

13. $3^2 > 2^3$

14. $2^4 - 1 < 4^2 - 2$

15. $(6 \times 2) \times (3 \times 2) = 18$

16. $2^3 < 2^4$

17. $(187 + 96) \times (187 + 4) = 18{,}700$

18. $4^{13} > 4^{12}$

19. $6^2 < 6^3$

20. $11^3 > 11^2$

21. $1^4 > 1^3$

22. ${}^+2 \times {}^+10 + {}^-2 \times {}^-10 = 0$

In the following expressions, choose a numeral to fill the frame which will make the problem as easy as possible to simplify mentally. Then mentally simplify the expression. Show your scratchwork in the space to the right if you find scratchwork necessary.

23. $8 \times 18 + \square \times 18 =$ ______

24. $93 \times 8 + \square \times 8 =$ ______

25. $\square \times 7 + 89 \times 7 =$ ______

26. $\frac{2}{3} \times 23 + \square \times 23 =$ ______

27. $93 \times 16 + 7 \times \square =$ ______

Do the following problems in the most simple manner possible. Show all your computation in the space below each problem.

28. If the temperature is $^{+}15°$ Fahrenheit and it drops 27°, what is the temperature at the end of the drop? ______

29. At the end of one day a store had a balance of $\$^{-}10.70$, and at the end of the next day the balance was \$15.30. How much better off was the store at the end of the second day than it was at the end of the first day? ______

30. Mrs. Jones was making a vegetable garden. She planted 17 rows of cabbages with 19 plants in each row, and 83 rows of tomatoes with 19 plants in each row. Altogether, how many plants did Mrs. Jones have in her garden (including cabbage and tomato plants)? ______

31. The highest temperature on January 10 in Salt Lake City was 5° above zero. On January 11 the highest temperature was 2° higher than on January 10. On January 12 it was 9° lower than on January 11. What was the highest temperature on January 12? ______

Scoring - cTT

Four of the thirty-one items were not used in the scoring. These included items 1, 5, 9 and 11.

Five of the remaining twenty-seven items were scored separately from the remainder of the items. These included items 4, 10, 15, 17 and 22. Five points were assigned for each correct response to these items, giving a range of 0 to 25 for this subtest. This subtest was denoted by cnTT.

The remaining twenty-two items were scored as follows:

Items 2, 3, 6, 7, 8, 12 and 30 received one point for the correct answer with an additional 4 points if the computational scratchwork evidenced that the problem was solved by use of the dpma.

Items 13, 14, 16, 18, 19, 20 and 21 received two points for each correct answer.

Items 23, 24, 25, 26 and 27 received two points if the choice was one of the easiest possible choices for applying the dpma, one point if the dpma was applied in a less simple manner, and no points if the dpma was not used, even if the answer was computed correctly some other way.

Items 28, 29 and 31 were given 5 points for each correct answer.

The summation of all the points gave a possible range on this segment of the test of 0 to 74. This segment constituted the main body of the test and was denoted by cTT.

Method of Administration - nsTT

(The examiner first hands a piece of scratch paper to every pupil and then reads aloud the following statement:)

We want to find out how quickly children can discover certain things. I have a rule in my mind. I want you to tell me a number. I will use my mental rule to do something to that number and will tell you the number I end up with. When you think you know my rule, raise your hand and I will let you take a turn in telling what the answer is. Don't speak out and give the rule away or tell it to anyone else. I will know that you know my rule if you can give the right answer. Let's keep the numbers you tell me smaller than 15 so I can use my rule easily on them. Who would like to try the first number?

(The examiner then selects pupils in turn to tell him various numbers. As each number is orally told to him, the examiner uses his rule to perform an operation on that number and then lists the results on the board in a table with the following form:

number	result
4	12
10	30
5	15
8	24

etc.

The first rule the examiner uses is that of multiplying by 3 each number told to him. The examiner continues to follow this process, allowing those children who discover the rule to help fill in the "result" column, though not allowing them to verbalize the rule. This continues until eight pairs of numbers have been listed on the board. The examiner then says:)

Now let me see what happens if I put a number in the "result" column. (Examiner places a number in the "result" column which is a multiple of three and which is not already listed in that column.) Do you think you can tell me what number would have to go in the "number" column in order to give us this result? (Examiner allows a child to give the answer. This process is repeated three times in all. The examiner then asks if someone would like to tell what his rule is. When the rule has been verbalized the examiner shows how it applies by checking each pair in the table. The examiner then follows the same procedure again, using a different rule, that of adding 11 to every number which is told to him. The examiner then hands out page A3.59 and says:)

Read the directions on your sheet silently while I read them aloud.

(The examiner reads aloud the directions and then answers any questions which give no hint to the rule used in the table. The pupils are then allowed to begin. When 85 per cent of the pupils have completed the page the papers are collected. The examiner then passes out page A3.60 and says:)

Read the directions on this page silently while I read them aloud.

(The examiner reads aloud the directions, stressing that the pupils make no marks on the paper after filling in the box. He then answers any procedural questions taking care to say nothing which will hint at the rule. The pupils are then allowed to begin. When approximately 85 per cent of the pupils have finished with all they can do, the examiner collects the papers.)

nsTT

On this page is a table similar to those we have been working on the board. You will notice that there are several blank spaces left in the table. Find out what rule is being used in this table and then fill in the blank spaces according to that rule.

number	result
2	4
3	9
4	16
1	1
6	____
8	64
5	____
____	81
10	____

All of you know what "odd numbers" are, don't you? Sure, they are numbers such as 1, 3, 5, 7, 9, etc.

PROBLEM - Find the sum of the first 30 odd numbers, without adding all of them together.

$$\underbrace{1 + 3 + 5 + 7 + 9 + \dots}_{\text{first 30 odd numbers}} = \boxed{}$$

In order to help you do this, we have constructed a series of problems below. We would like you to begin working them. Fill the blanks on every problem you do so we can see how far you work. When you believe that you know the answer to the problem above, you may put the answer in the box and stop working on the problems below. Do not work any problems below after you put the answer in the box.

1 + 3 = ______ The sum of the first 2 odd numbers = ______

1 + 3 + 5 = ______ " " " " " 3 " " = ______

1 + 3 + 5 + 7 = ______ " " " " " 4 " " = ______

1 + 3 + 5 + 7 + 9 = ______ " " " " " 5 " " = ______

1 + 3 + 5 + 7 + 9 + 11 = ______ " " " " " 6 " " = ______

1 + 3 + 5 + 7 + 9 + 11 + 13 = ______

The sum of the first 7 odd numbers = ______

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = ______

The sum of the first 8 odd numbers = ______

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = ______

The sum of the first 9 odd numbers = ______

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = ______

The sum of the first 10 odd numbers = ______

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 = ______

The sum of the first 12 odd numbers = ______

Scoring - nsTT

On page A3.59, 5 points were given for each correct response, giving a possible range of 0 to 20. The correct responses were, in order from top to bottom: 36, 25, 9 and 100.

On page A3.60, 20 points were given to each person who placed the correct response (900) in the box. One point was subtracted from this score for each of the series problems which were worked prior to making the discovery. For example, if a pupil responded correctly after filling in 7 of the series problems, he would receive a score of 13 (20-7).

If a pupil failed to place the correct response in the box, he received a score of 0.

This scoring system gave a possible range of scores from 0 to 20 with no scores assigned between 0 and 10.

The total points on pages A3.59 and A3.60 were summed together to give a total nsTT score. This gave a possible range of 0 to 40 for the total test.

Method of Administration - apTT

(The examiner uses the blackboard to administer the first part of this test. After each pupil is provided with scratch paper the examiner says:)

Today we are going to do some computing on the board. If any of you think that you discover anything about these problems, please keep it to yourself. Do not tell anyone else how you are working these problems. Let's begin.

(The examiner then places on the board the series of problems on page A3.64, letting the pupils respond to each one in turn and checking their answers by working within the parentheses first as if oblivious of the associative principles which could be used. After three problems have been worked the examiner asks the pupils to raise their hands as soon as they have the answer. As soon as two or more children have evidenced by their speed that they have discovered the shortcut the examiner should exclaim:)

My, you (add/multiply) quickly (John). I don't believe I could possibly have worked the problem that rapidly.

(The examiner then continues, making certain that no pupil reveals the shortcut to any other pupil. When the examiner reaches problem 23, he selects a pupil who is obviously working the problem by the shortcut. He asks this student to give the answer. When the pupil gives the correct answer the examiner says:)

Certainly (John), and we know that answer is correct because 187 x 25 equals what? John? Well John, don't you know? You just gave me the right answer. You must be working the problems some other way!

(The examiner then continues, not allowing John to verbalize his shortcut. When all 30 problems have been placed on the board - always leaving as many as possible on at the same time, the examiner erases the board, passes out page A3.65, and says :)

On this page there are six problems of the type we have been doing on the board. I want you to do these problems as you have been, except this time I would like you to show your work on paper. In the space to the right of each problem is a space in which I would like you to show every computation, even those which you normally do in your head. Then place your answer in the blank after each problem. You may begin.

(No time limit is imposed. Each student is allowed to complete as much as he is able to do.)

Scoring - apTT

One point was given for each correct answer, with an additional four points if they solved it by use of the appropriate associative principle (as evidenced by the computational scratchwork).

Possible range - 0 to 30.

A3.64

1. (13 x 5) x 2 = ______

2. (48 + 7) + 3 = ______

3. (53 x 2) x 5 = ______

4. (9 + 5) + 15 = ______

5. (98 x 25) x 4 = ______

6. (161 + 89) + 11 = ______

7. 2 x (5 x 27) = ______

8. 975 + (25 + 983) = ______

9. (687 x 20) x 5 = ______

10. 13 + (17 + 9) = ______

11. (37 x 5) x 2 = ______

12. (87 + 9) + 91 = ______

13. 20 x (5 x 84) = ______

14. 143 + (7 + 100) = ______

15. (17 x 20) x 5 = ______

16. $(2\frac{1}{2} + 1\frac{1}{4}) + \frac{3}{4}$ = ______

17. (897 x 4) x 25 = ______

18. (387 + 9) + 1 = ______

19. 5 x ($\frac{1}{5}$ x 634) = ______

20. (583 + 92) + 8 = ______

21. 2 x (50 x 63) = ______

22. (82 + 49) + 51 = ______

23. (187 x 25) x 4 = ______

24. 13 + (987 + 5426) = ______

25. (7 x 25) x 2 = ______

26. 25 + (75 + 7452) = ______

27. (12 x 15) x 2 = ______

28. 15 + (85 + 38) = ______

29. (73 x 81) x 0 = ______

30. (23 + 88) + 12 = ______

Start here:

1. (382 + 925) + 75 = ____

2. (296 x 25) x 4 = ____

3. 14 + (86 + 78) = ____

4. (716 x 50) x 2 = ____

5. (87 + 97) + 3 = ____

6. 2 x (5 x 439) = ____

olR - Rating Scale - taR*

Teacher Name: ____________________ Rater: ____________________

Date: ____________________

Pages: ____________________

Discovery - Expository Items:	(Circle One)
A. Teacher omniscience	0 1 2 3 4 5
B. Introduction of generalization	0 1 2 3 4 5
C. Method of answering questions	0 1 2 3 4 5
D. Control of pupil interaction	0 1 2 3 4 5
E. Method of eliminating false concepts	0 1 2 3 4 5

General Items:	
F. Correct use of materials	0 1 2 3 4 5
G. Well-prepared and well-organized	0 1 2 3 4 5
H. Enthusiasm	0 1 2 3 4 5
I. Non-evaluative climate	0 1 2 3 4 5
J. Checks for student understanding	0 1 2 3 4 5
K. Uses student responses	0 1 2 3 4 5

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*This instrument was adapted from Latham, Marilyn, & Worthen, B. R. "Rating Teachers on the Discovery-Expository Dimension of Teaching," unpublished manuscript, University of Utah, 1965. (Mimeo)

Use and Scoring - olR, taR

Items A through E are used to differentiate between discovery and expository teaching behaviors. A rating of 5 indicates the greatest degree of discovery teaching behavior. A rating of 1 indicates the greatest degree of expository teaching. A rating of 0 indicates that the item was unobserved or had no occasion to occur.

Rationale for the rating of each item follows.

A. Teacher omniscience (rate either 1, 2, 3, 4, or 5 as indicated below)

1. Teacher certainty. The teacher acts as the primary source of knowledge concerning arithmetic. The students may depend on the teacher when they cannot work a problem. The teacher always indicates that he will show the student how to work the problem correctly. He is never doubtful or uncertain. The teacher checks the answers of the students and shows them how the correct answer is obtained by use of the principle involved. When an incorrect answer is given, the teacher recognizes it and immediately asks the student if he is certain that his answer is correct. This gives the student an opportunity to correct his own mistake. If the student is unable to do so, the teacher asks for someone else in the class to respond. Care is taken, however, to avoid any negative evaluation of the student's response.

2. Exhibits many characteristics of 1, but also some characteristics of 5.
3. Exhibits characteristics of 1 and 5 equally.
4. Exhibits many characteristics of 5, but also some characteristics of 1.
5. Teacher uncertainty. The teacher does not act as the primary source of knowledge concerning arithmetic, but seems to depend upon the students to help him work the problems. He sometimes reflects an uncertain feeling about the precise way to solve a particular problem. When given an answer, whether correct or incorrect, he checks it by the long method as if he is not aware of the principle which would allow for solution by a "shortcut." When a student gives an incorrect answer, the teacher continues on to the next problem as if unaware that the answer is incorrect. When a student points it out, the teacher acts surprised. If the students fail to notice the mistake, the teacher goes back a short time later, as if he has just noticed it, and questions the correctness of the earlier response. The student who gave the response is allowed an opportunity to correct it. If he is unable to do so, other members of the class are asked to respond.

B. Introduction of generalization

1. The teacher gives the generalization (rule) before the students are given any examples. All examples are then related back to the rule for solution.

2. The teacher explains the rule simultaneously with explanation of examples. The generalization is not made completely clear before the examples are introduced but is clarified by use of the examples.
3. The teacher generalizes when approximately one-half of the students have become aware of the principle through working the examples. It is much too soon to generalize if the teacher is teaching by the discovery method and much too late if teaching by exposition.
4. The teacher delays the generalization until many of the students have discovered it through working examples but states the rule before some of the students have become aware of it.
5. The teacher delays the verbalization of the generalization until all, or nearly all of the students have made the discovery. He is careful to give no hint that there is a "shortcut" to working the problems. He also takes care to avoid the use of vocabulary terms related to the generalization.

C. Method of answering questions

1. The teacher answers questions by reiterating and explaining the rule and relating it to the question. The teacher then gives examples which will further clarify the way the rule is used in the solution of that type of problem.
2. Exhibits many characteristics of 1, but also some characteristics of 5.

3. Exhibits characteristics of 1 and 5 equally.
4. Exhibits many characteristics of 5, but also some characteristics of 1.
5. The teacher answers questions by referring to the model or the computational sequence which the student has used. If a student is still confused, the teacher takes him back to the model and goes through it carefully. The teacher may make use of sequenced examples as a clue, but no verbal hint of the rule is given.

D. Control of pupil interaction

1. The teacher allows the students to share ideas about arithmetic. He encourages them to help each other if they can show the other person how to apply the rule. Often he allows them to check an answer with their neighbor but insists that they do their own work.
2. Exhibits many characteristics of 1, but also some characteristics of 5.
3. Exhibits characteristics of 1 and 5 equally.
4. Exhibits many characteristics of 5, but also some characteristics of 1.
5. The teacher always prevents the students from sharing ideas about arithmetic. The teacher insists that no student does anything which may inhibit another child's discovery by giving the rule to him prematurely. If a student does verbalize the rule, the teacher is displeased.

E. Method of eliminating false concepts

1. The teacher warns the students of common errors made in applying the principle. He points out specifically the types of problems the students are likely to make errors on and then gives examples of each kind of error.
2. The teacher warns the students that the rule will not work on all of the problems on a certain page. He informs the students that there are some "trap questions" and suggests that they check each problem carefully to see if it fits the rule.
3. The teacher warns the students to watch for false generalizations in general and to check carefully before using the rule in any situation. He includes "trap questions" to reinforce his warning.
4. The teacher includes "trap questions" but does not warn the student to be careful in applying the rule. He merely tells the student to be careful in solving the problems. No reference is made to the rule.
5. The teacher includes "trap questions" and gives no verbal warning of any type. The teacher purposely leads the child, through sequencing of examples, into overgeneralizing the rule. If the problem is missed, the teacher waits until the error is mentioned, then checks to make certain it is wrong. He says nothing about the rule or why the answer was incorrect.

Items F through K are used to rate the teacher on general classroom teaching. A rating of 5 indicates the highest possible rating on each item. A rating of 1 indicates the lowest possible rating. A rating of 0 indicates that the item was unobserved or had no occasion to occur.

Brief explanation of what is intended in each item follows:

F. Correct use of materials. In order to retain purity of treatments, the materials must be used correctly. This means that the teacher uses the materials in the proper sequence, asking the appropriate questions and presenting the appropriate explanation at the specified time. No deviation from standardized procedure is allowed.

G. Well-prepared and well-organized. This may seem self-explanatory, but should nonetheless be noted carefully by the rater. For example, if a teacher does not have the materials ready to distribute to the students at the beginning of class, this would result in a low rating.

H. Enthusiasm. Enthusiasm includes the projection of a positive reaction toward arithmetic and toward the method of teaching the teacher is using. It can, however, be overdone. If a teacher is assumed to be overly or insincerely enthusiastic, his behavior will be rated "2" on the scale.

I. Non-evaluative climate. The teacher does not seem to attach a value judgment to his response to a student but establishes a climate where the student feels free to respond even when he is not certain of his answer. The teacher does not evaluate negatively or in any way which offends the student or puts him on the defensive. The teacher creates a feeling that everyone makes mistakes and no stigma is attached to being in error. An attempt to establish a "non-evaluative" climate can be overdone, however. If the teacher praises the students so much as to appear insincere, he will be rated "2" on the scale.

J. Checks for student understanding. The teacher must make frequent checks to see that the students understand what they are doing.

In the discovery method, the teacher checks by getting individual responses (lip reading, walking around to check each answer) or by the use of test questions which are so difficult that they could only be answered quickly by those who have made the discovery.

In the expository method, the teacher checks understanding by asking questions and checking individual responses. He may walk around in order to check each answer and see that the students are applying the rule correctly.

K. Uses student responses. The teacher allows the students to respond whenever possible. He gets student responses to problems on the board and also asks oral questions to get responses. He allows students to correct each other. In the discovery method, this means letting the students catch the errors. In the expository method, it means letting students show the rest of the class the correct way to work the problems.

APPENDIX IV

Miscellaneous Materials

SALT LAKE CITY PUBLIC SCHOOLS

January 11, 1964

Dear Parents:

In cooperation with the Salt Lake City School District, the University of Utah Educational Psychology Department will be conducting a research project in your child's school during the next few weeks. This project is being conducted for the purpose of determining more effective ways of teaching mathematics to elementary school children. This effort is being financed by a grant from the United States Office of Health, Education, and Welfare. Dr. Gabriel Della-Piana is director and Blaine R. Worthen is supervisor of this project.

This project will be conducted in nine elementary schools in the Salt Lake City District. These schools were selected from among all of the schools in the Salt Lake City School District upon the basis of two criteria: (1) The possibility of the necessary administrative details being worked out in this school, and (2) the existence of a teacher in this school who was willing to cooperate and who possessed the requisite qualifications to work on this project.

Your child's math teacher has been selected and has prepared for this project by attending a university class in mathematics and research and teaching techniques. The choice of the teacher has determined the choice of the groups which will be involved in this study. Your child's class has not been selected because of any special ability.

This project will run for approximately six weeks with tentative dates now set of January 11th to February 26th. During this period of time, there are several unusual conditions which must exist due to research procedure. We sincerely hope that you will be willing to cooperate by helping us maintain these conditions as the very success or failure of this project may well hinge upon these conditions being met. During the six weeks of the study, the following changes from the regular procedures will be made:

1. During the six weeks of the study, there will be absolutely no arithmetic homework assigned to your child. All of his arithmetic will be conducted during school hours. This is to insure that all children involved in the project will receive only the exposure to arithmetic which they receive

in the carefully controlled classroom situation. In line with this, we would ask that you avoid working with your child at home on arithmetic during this period of time. It is our considered opinion that your child will not suffer detrimental effects by following this procedure.

2. Your child will not be allowed to bring any of the materials home which he is working with during this six-week period. This is not intended to in any way be secretive but is again due to suggested research procedure. More than one class in each school will be involved in this study, and it is essential that any interchange of ideas be precluded. After the six-week period is over, all of the materials your child has been working with will be sent home for your examination.

3. In the publication of the results of this study, individual children will never be identified by name nor will they be compared with one another in any way other than that which would be normally expected. Their performance throughout this six-week period will, however, in all probability, account for a segment of their final grade in arithmetic.

If this memorandum has not answered any questions which you may have in your mind, I am sure that your child's teacher will be happy to discuss them with you. We appreciate your cooperation, and it is our hope that the work of these teachers and the information that we thus obtain will contribute to better education for tomorrow's students.

Sincerely yours,

SALT LAKE CITY SCHOOL DISTRICT

James O. Morton

James O. Morton
Director of Elementary
Curriculum and Instruction

UNIVERSITY OF UTAH

Gabriel Della-Piana

Gabriel Della-Piana
Project Director

Blaine R. Worthen

Blaine R. Worthen
Project Supervisor

INTRODUCTORY STATEMENT COMMON TO TREATMENTS D AND E

A group of mathematicians and teachers at the University of Utah have been working the last two years on trying to find out more about teaching arithmetic to elementary school children. They are interested in finding better ways to teach mathematics, particularly at the fifth- and sixth-grade levels. In order to find out more about the teaching of arithmetic, they have written some special materials and have used them at the University of Utah with small groups of children. They are still not certain, however, if these materials are really very clear and easy to understand, and so they have asked us to help them to find out how clear they are. So we will use their materials for the next few weeks. This will give them a chance to find out what works best in their materials. They have written these materials in two different ways. For the most part, they are very similar, but there are some slight differences between the two sets. For this reason, it is very important that you do not discuss these materials with anyone outside of this classroom until we have finished using them. This includes your parents, your brothers or sisters, and particularly anyone else in this school. After we have used these materials and have finished in a few weeks, you will be allowed to take all of the materials home to keep, and to share them as much as you like with each other, with people in other classes, or with your brothers and sisters and parents. But let me repeat, you should not, under any circumstances, talk about them outside of this class or show them to anyone outside of this class until we are through using them a few weeks from now.

Of course, we cannot police you at all times, nor do we really want to be in the position of having to check you to make sure that you are doing this as we have asked. Instead, we feel we can trust you. We are certain that you people in the fifth and/or sixth grade are old enough and mature enough to be able to really help us in this study. I am certain that you realize that you could torpedo the research study that these people at the University of Utah are trying to do by merely telling other people outside of our class about what we are doing before we are allowed to do so. We are sure that you will feel, however, that there is no useful purpose to be served by thus hindering the work of these people. They have worked hard to produce these materials and now we can help them decide whether or not their materials are useful. By helping them, this may help us sometime in the future. Therefore, we are placing our trust in each of you to restrain yourself from sharing these materials with anyone outside of our class. (The remainder of this paragraph was given only to students in Treatment D.) In addition, there are many times, in fact the majority of instances, in which you will be asked not to share these materials or talk about them with members within our own class. Again, at a later time you will be able to talk about them in our total class and discuss them freely then.

There may be some special times when you will be allowed to take work home for homework. You will have to try particularly hard at these times to refrain from sharing these materials or getting help on them from your brothers and sisters or parents.

Along with these materials, you will be asked to take several different tests, once before we begin using the materials, and again when we have finished. These tests will tell us many things about the materials. "Tests" may not be the best word, because many of these will merely be forms which you will fill out to tell us how you feel about certain things that have to do with school. None of these tests will be used in any way to grade you, with one single exception. There is one test that will be given to you at the end of the six weeks that may be used to make up a small part of your arithmetic grade. This test will not count heavily upon your total grade, however. Otherwise, none of the tests will be used in grading you. We are merely interested in finding out more about the materials. The only way that you could do poorly or fail these tests is to not be honest in your replies to any questions which you are asked. Lack of honesty on your part would probably show up when the tests are scored. I want you to understand that as your teacher, I honestly do not care how you respond on any of these tests, as long as you respond honestly and to the best of your ability.

Do you have any questions?

Sample Transcription of Audio-taped Lesson*

Teacher 1
Treatment E
Lesson #1

Teacher: Boys and girls, ever since you have been in school, you have been given numbers to work with, and a very fine set of numbers. You can do many things with them. Let's call these numbers of arithmetic. With these numbers of arithmetic you can go 4 times 8 equals 32, . . . you can do many things with them. But there are some things that we can't possibly do with them, such as problems of 0 minus 3. We can't work that with our numbers of arithmetic. 10 minus 12? We can't work that one. And there are others that we can't work. Today I'm going to introduce you to a new set of numbers, numbers that will permit you to do problems such as 0 minus 20 or 30. The best way for me to show you the necessity of these numbers is to give you a physical model. And I want you to watch, while I draw the road. (Draws road.) This is a road and it has markers on it that are all letters of the alphabet. A little different than the markers that we usually see. The markers on this road are exactly 1 mile apart. . . . On these signs we have letters, for instance, E here, L here, Y here, K here, X here. . . . There is 1 mile between each of these signs. Now how far would it be if we were to go from K to M? How far would it be?

Pupil: 3 miles?

Teacher: 3 miles. Starting here, we go 1, 2, 3. Now if we go from X to Y, tell me how many miles, how far it is from X to Y.

Pupil: 3 miles.

Teacher: 3 miles from X to Y. Suppose we go from P to A. How far will it be?

Pupil: . . .

Teacher: 7 miles from P to A. Suppose you are at point T on this road and you decide to take a trip to R. Karen? Pardon?

Karen: 5 miles.

Teacher: Karen says 5 miles. Well, no, Karen, that isn't quite right. Would you check it again?

---

*Ellipses are used when a brief section of the recording could not be heard distinctly.

(Part of tape was so faint it could not be transcribed. . .)

Teacher: If you and your friend each took a 2-mile trip and you go 2 miles to B, then your friend would go . . . and you would be 4 miles apart. 1, 2, 3, 4 miles apart. If you took a 2-mile trip over to Z and your friend took a 2-mile trip over to Z then you would be right together. If you and your friend each took a 2-mile trip then you could have two answers. You could be right together or you could be 4 miles apart. So we have to have something more to tell us how to work certain problems. These numbers of arithmetic are fine to use, but we have to have something that really tells us. So we're going to call these real numbers. The numbers you have used have been called numbers of arithmetic. But these numbers that tell us what we really need to know are real numbers because these really tell us. Now mathematicians have worked out a way of doing this. This number, positive 3, means, to the right. This number would mean to the -- you tell me, Jeff?

Jeff: Left.

Teacher: Left or negative. So now let's go back to our road. Our road would show this way then. This is positive and over here, this is negative. (indicates directions) Positive and negative. Now, I've said that if I'm at L and I'm going to take a positive 3 trip, we can call it a positive 3 trip, we would definitely know where I would be. Where would I be, Chris? Z. 3 miles would put me at M because it tells me exactly which way I am going. So the mathematicians have figured out these real numbers to show us which way to go on our road. This is the positive and this is the negative.

(unable to hear part of the tape)

Teacher: . . . of arithmetic, that you have used all this time, will tell you and show you the difference. It will still show the difference. What we lack and what we need will be the direction - whether we're going this way (points) or whether we're going this way. (points) So our numbers of arithmetic will easily tell us how far we are going, or the distance; but what we need is something more that will tell us the direction, so that we have both of these to get one answer. Now the mathematicians have used what we call a positive and negative. This way would be your positive direction (points) and this way would be your negative direction, (points) or right and left. Now our mathematicians give us the correct pronunciation of it and it would be positive and negative. So then we would call this a what?. Read this for me, Karen.

Karen: Positive 2.

Teacher: Positive 2. Read this one, Colleen?

Colleen: Negative 2.

Teacher: And if I said positive 2 miles, point which way you would go. (class points) If I said negative 3 miles which way would you go? (class points) And so this gives us our direction. Now, you have 4 sheets on your desk. Sometime later I would like to have you read through the first three, but right now will you just turn the bottom one over. Look at the corner and see if it says A2.05. That's the one I would like to have you see. Here is a road that shows your negative and your positive. Everyone just turn over A2.05 please. Let's look carefully at this road. It shows us our negative and our positive. If I were at V, and I wanted to take a trip of positive 3, where would I end? L. If I were at V and I wanted to take a trip of negative 5, where would I be? F. How many agree? (some raise hands) How many disagree? (a few raise hands) Where do you think I'd be, Dale?

Dale: B.

Teacher: V, and 5. Count it over. 1, 2, 3, 4, 5. You didn't count quite far enough. We can't count from where we're starting from, can we? If I was going to go downtown 3 miles, I couldn't start with 1 mile because I haven't gone anywhere, have I? I'm still here. If I were going 3 miles, I'd think, well, here's 1 mile, here's 2 miles, 3 miles. I haven't even gone anywhere, so this would be 0, wouldn't it? After you go 1 mile, you count it. Let's see what you can do with these examples now. The example shows you K to T. The solution shows you that that's a positive 3. Look up at your road and see if that's right. K to T. Is that right? (children respond affirmatively) See if you can fill in the real numbers for those sets. Let's do it quickly.

Sample Transcription of Audio-taped lesson*

Teacher 1
Treatment D
Lesson #1

Teacher: Boys and girls, let's imagine we're on a road. (draws road) It's a little bit crooked isn't it? On this road are signposts. (draws signposts) You boys that came in, we're supposing and imagining we have a road, and on this road there are signposts and each signpost is 1 mile apart. Now these signposts are marked a little differently than ordinary signposts. Each one has a letter. Now each one of these signposts, as I said, is 1 mile apart. Now supposing that you are at Y and you decide to take a 3-mile trip. Raise your hand if you know where you would be at the end of your 3-mile trip. Michael?

Michael: At E.

Teacher: At E. How many agree that you would be at E? (several raise hands) Does anyone disagree? (three students raise hands) Christine? Steve? O.K. let's check it, shall we? We're taking a 3-mile trip; Michael said we would be at E. How would you be at E, Michael?

Michael: Well . . .

Teacher: Let's check it. 1, 2, 3, right? Christine thinks we'll be somewhere else. Let's see if she will be. Where would you go, Christine?

Christine: . . .

Teacher: She's at Y and she goes 1, 2, 3. How many think Christine's right also? Now let's suppose that you and your friend each decide to take a trip of 2 miles. At the end of the trip how far apart would you be? Paul?

Paul: 4 miles.

Teacher: 4 miles. Let's check it, should we? We start at Y, where are you going, Paul? Where would your friend be?

Paul: X.

Teacher: X. How could he get at X?

Paul: He would be going the other way.

---

*Ellipses are used when a brief section of the recording could not be heard distinctly.

Teacher: Edith?

Edith: . . .

Teacher: How could they both go the same way? They're both here. What do you mean? . . . Why? . . .

Michael: . . .

Teacher: Right! They could both go to X. They could both go either way, couldn't they? Now boys and girls, it's easy to see that we cannot give one definite answer, one correct answer for these questions. Now ever since you have been in first grade or in school, you have been using numbers. Numbers such as 4, 7, 8, etc., and these are, let's call them numbers of arithmetic. Now these numbers of arithmetic tell what about our trip? They tell us the what? Dennis? The distance. These numbers of arithmetic will tell us the distance. 3 miles, 2 miles. But now to get a definite answer we need something else. What do we need? Not only the distance but the what? Tracey?

Tracey: Which way we're going.

Teacher: Which way we're going, and what's a good word for that? Michael?

Michael: Direction.

Teacher: Direction. So our numbers of arithmetic give us the distance, but we need numbers that will give us both the what? The distance and the direction. If they'll really tell, let's call them real numbers. Now if I'm going to have a distance of 3 what kind of a sign might I put on to show that I'm going that way? (points) What little sign could I put, what little symbol?

Pupil: An arrow.

Teacher: An arrow. That might work just fine. That would tell us definitely - what would that really tell us? Stephen? The direction, and so how would you read this? This would tell us that we're going East, or, Corey?

Corey: West?

Teacher: Well, it would have to be. Michael?

Michael: . . .

Teacher: Well, but what direction? Dennis?

Dennis: . . .

Teacher: Well, of course the top of our map is always north but we're situated a little differently. But instead of using east and west, what other kind of direction could we use? Janey?

Janey: Right.

Teacher: Right, couldn't we. Then it wouldn't matter where we were located. It's a little confusing because this is east and that's west, but if we put the top of our map right here this would be north and this would be east. Right? So let's use the word, right. O. K., so who could read this for me then? We'd call this what?

Pupil: Right 3?

Teacher: Right 3. We wouldn't necessarily have to have the miles because we might be going 6 maybe. So then this would be called, what would it be called, Tracey?

Tracey: Right 3.

Teacher: Right 3. Do you think this might be called a real number? Now let's see, you're going to take that trip, beginning at C. Where would you end up? Where would you end up, you're beginning at C? Where would you end? Betty?

Betty: At B.

Teacher: At B. Let's check shall we? Betty's here, she's going 5 miles. That's 1, 2, 3, 4, 5. Does that put you to B? Paul?

Paul: . . .

Teacher: Well, we might have a little difficulty with that because we might go different directions. Let's wait and see, shall we? Now I'm going to give you a check. Alright, what's this? Right 9. Alright, 1, 2, 3, 4, 5, (goes off end of road) that one wouldn't work. Dead end. Wait! I have to figure out where I am. Let's check it. What's this one? Right 2?

Pupil: Right 1.

Teacher: Right 1. So you would end up here, wouldn't you? That's where you would end up. Now, turn your last sheet over, the last one you have, turn it over. Look in the corner and see if you have A1.05. Let's look at the road. Let us agree that in this exercise, the number, what shall we call that? Janey? Right 3, measures a 3-mile-to-the-east trip, or to the what? To the right. Then what's the next one, Edith?

Edith: Left 3.

Teacher: Now let's have everyone try number 2 down here. See how well you can do on it. Who could give me three trips for that? Corey?

Corey: . . .

Teacher: Alright, let's check it. A to X. Here's our A and we're going what, right 2? 1, 2. O.K. L to M she says. I wonder about that. Alright. P to Q. (circulates) Some boys and girls only put 1 trip. Maybe you didn't read the top part. It asks you to put three trips. Now let's look at C. Imagine taking trips along an east, west road or right, left road and trips to the east or the right are measured by right real numbers, using the following . . . use the following measurement of trips to fill the markers correctly. In this road at the bottom are instructions of the trips and I'm going to put more on the board, because we didn't get all of them in at the bottom of the page. So after you've put more on your paper put down those on the board, follow those on the board. Read it quickly and let's have everyone start and let [illegible] see what you can do.

Sample Transcription of Audio-taped Lesson*

Teacher 8
Treatment E
Lesson #2 - excerpts

Teacher: We're going to review, now, the rules in connection with the work we've been doing with real numbers.

(teacher reviews and explains rules)

Teacher: Now we're well armed. Are we ready to check? We have finished the checking; we finished it yesterday, A2.08 to A2.10. Right? Will you now open your folders to page A2.11? Will you take a moment or two to glance over these and then we will begin doing a few. (pause) Problem 7, Corey?

Corey: Positive 8, plus negative 6, equals positive 2.

Don: Negative 1,000 . . .

Teacher: That was problem 9. Let's do problem 8 now, Karen, please?

Karen: Positive 11, plus negative 4, equals positive 7.

Teacher: Next?

Pupil: Positive 7, plus negative 1, equals plus 6.

Teacher: Sure?

Pupil: Negative 6?

Teacher: What's the rule on this?

Pupil: Oh, positive 6.

Teacher: Why?

Pupil: Because you minus 1 from . . . well, positive seven from negative 1 is positive 6.

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*Ellipses are used when a brief section of the recording could not be heard distinctly.

Teacher: You're showing the difference, but you were concerned about whether the six was positive or negative? Why is this the way you said it was just now?

Pupil: Well, because you . . . well, because . . . well, you take the negative . . . you take the minus, well you take the negative away from the positive in that rule.

Teacher: In that rule. And then the rule goes on to say that our answer, our sum, takes the symbol as a sign of which one of our real numbers? . . . Because it's larger, isn't it? This is what you were trying to say, wasn't it, Chris? O.K. Let's go on to number 12, please. Mark?

Mark: Negative 15, plus positive 13, equals negative 2.

Teacher: We're a little mixed up on our numbers. The girls are telling me we omitted number 10. Ten was just taken care of so now we're on number 12. Are we on the right track now? Number 12, here we go.

Pupil: . . .

Teacher: Mark, I'm sorry, David.

David: Positive 6, plus negative 4, equals positive 2.

Pupil: Positive 7, plus negative 5, equals positive 2.

Teacher: Next? Number 20 . . . 15, I'm sorry.

Pupil: . . .

Teacher: There's the first one that she was ready for. 16? Are you ready Janet, too?

Janet: . . . Ummmm.

Teacher: We'll come back to you, dear, so you'll be more accustomed because of your absence.

Pupil: . . .

Teacher: Are you sure? Chris, can you help her out?

Chris: It's about like mine. You'd have to take the negative 19 away from the positive 20, leaving a positive 1.

Teacher: Now, your hand is up, Joanna. Why? Because it's a larger number. That was good, except we couldn't quite hear you in the front. Now, let's go on using the rule from 3 to 5, which involves again our positive and negative numbers and what we're going to do with them. Are you ready for us, Debbie? We're on number 1.

Debbie: . . .

(teacher continues to elicit solutions to problems from children)

Teacher: Alright, we're questioning number 14. What about the symbols of your real numbers? Negative 8, and what kind of a 9? Positive 9. So we should do what? Cheryl, would you like to give us the information on that?

Cheryl: . . .

Teacher: Because that is the little rule that we're following, is it not? So what would the answer be now, Cindy? Positive 1. Very good.

(teacher continues to elicit solutions to problems from children)

Teacher: How do we feel? Are these rules a part of us so we can think back and recall them? . . . By way of summary, let's check back for a moment because of those people who have been absent and see if we can clear a couple of rather important details. They're not at all minor. We are being disturbed in terms of like symbols for our real numbers and what we do with them. Will you review rule 1 in connection with finding the sum of two real numbers with positive signs? How about Sharon.

Sharon: You'd add them.

Teacher: Fine. Let's say that if we have real numbers with negative signs, what are we doing?

Pupil: You could still add them, but your answer would be with a negative sign.

Teacher: Anyone now that doubts this? Let's then check this other part.

(teacher continues checking with students)

Sample Transcription of Audio-taped Lesson*

Teacher 7
Treatment D
Lesson #2 - excerpts

Teacher: Let's go back to our road and see if we can measure trips one after another. All right, supposing you are at point A, I'm giving you a trip, I'm giving you this trip. Who can read it? Howard?

Howard: Right 3.

Teacher: Right 3. Where would you be at the end of your trip, Howard? Good, three spaces to the right, so that'd be 1, 2, 3. Is that where you'd be? You'd be at B. Now, I'm giving you this trip. Where would you be? Where would you be on this second trip, Peter? Read it for me.

Peter: Right 2.

Teacher: Right 2, and where would you be? Two spaces to the right of what? Would you be right there? Alright, how many think that he would be right there? (some hands go up) Do you think he would be somewhere else, Bill? Yes, would he be two spaces after B? Wait, you were already at B, and I didn't put you back at A, did I? No. So this trip would be where? Bill?

Bill: . . .

Teacher: 2 right. 1, 2, would you be there? Now, let's check it out. You were already at A. This trip is a what, class? And there's 1, 2, 3, you'd be at B. And then you're going, 1, 2, and you're at C. Now who can tell me, from the beginning trip to the very end, what single trip would have taken you the same distance. Robin?

Robin: Right 5.

Teacher: Right 5, she says, so let's check it. Let's see if that would equal right 5. Count it. (with class) 1, 2, 3, 4, 5. That would equal right 5, wouldn't it? That single trip would have covered the distance, measured the distance. Let's try another one. You are at A. I'm going to give you this trip. Where would you be? Count them. (with class) 1, 2, 3. Where would you be, Galen?

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*Ellipses are used when a brief section of the recording could not be heard distinctly.

Galen: 3 spaces right of A.

Teacher: 3 spaces right of A. You would be at B. Where would you be now, Debbie? From what point? From B. Five spaces from B, let's count them 1, 2, 3, 4, 5. Now count it out and tell me, from the beginning point of A to the point of C, what single trip would have covered or measured the distance. Robin?

Robin: Left 8.

Teacher: Left 8. Let's try it. From A you go left 8. 1, 2, (student interrupts) no, she says. Marty?

Marty: Left 2.

Teacher: Left 2, she says. We start at A, left 2; 1, 2. That would put us there. So then that one single trip of left 2 would have taken us the same distance. Bill?

Bill: . . .

Teacher: No, one single trip that would take you from the beginning to the end. Let's try one more. Alright, we are at point A, this is your first trip. Where would you be at the end of the trip, Gloria?

(teacher continues to present examples)

Teacher: Alright, left 8. Somewhere, from which point? B. 1, 2, 3, 4, 5, 6, 7, 8. Now, what single trip would . . . I beg your pardon, Kristi?

Kristi: Left 3.

Teacher: Left 3. You're at A, you go 1, 2, 3, that would be left 3. I have some papers I would like to give out to you so we'll take a minute to do that.

Teacher: This morning we have taken, or measured, one trip after another. On your paper there are some trips for you to guide. I've also given you a sheet of paper. Now on this paper, you may draw your rows and measure your trips, either right or left, and when you are through measuring, please write down the single trip you could have taken, or the single direct trip that would have measured the overall trip. And here's your paper, you make rows and rows and rows, and even on the other side. And if you need more paper, you may have it. So please draw your rows, figure out your trip, and then tell me what single trip would have measured it. Let's not ask any questions now.

(after short time lapse)

Teacher: Let's do one or two more trips on the board. Some of you are getting a little lost on your roads and you'll have to watch carefully so you can get the right answers. Some of you are working very well and have your road fine; some of you are using the same road over and over. So we must be very careful that we get the road listed right. Alright, I am at A. My first trip is going to be this. What is it, Marty?

Marty: Right 5. No, left 5.

Teacher: Left 5, so 1, 2, 3, 4, 5, and we will be at B. Now my next trip is going to be left 2. Where would I be then? David?

David: . . .

Teacher: Which way? Alright, then I'd be 1, 2 - I'd be here at C. Now what single trip would measure the difference from the beginning to the end. Gloria?

Gloria: Left 7

Teacher: Left 7. So then 1 single trip of left 7 would be the single trip that would take me there. Let's check it out again. I am at A. The first trip would be left 5. 1, 2, 3, 4, 5. I am at B. My next trip would be at what? Bart? Left 2. 1, 2, I am at C. Now let's count out and see what single trip would take me there, ready? (with class) 1, 2, 3, 4, 5, 6, 7. So left 7 would have taken me there. Let's try one more and then our time will be about up. Our first trip is going to be . . . what is it? Right 3, so where would I be?

Debbie: . . .

Teacher: 1, 2, 3, I am at B. My next trip will be, Mary Jane?

Mary Jane: Left 5.

Teacher: Left 5. Where would I go then, Mary Jane? . . . From which point? (student points) Alright, let's count it. 1, 2, 3, 4, 5. Now, what single trip would take me the same measured distance? David?

David: . . .

Teacher 3 left, he says. Did I correct it, David? 3 left, he says. Let's check it from A. Left 1, 2, I would be at C. So your trip of right 3, left 5 could be measured with what? O. K. Watch your road carefully. We'll take about 5 more minutes for you people to draw your roads and finish up these 21 problems.

APPENDICES TO THE EXPERIMENT ON "DISCOVERY VS. EXPOSITORY SEQUENCING IN A PROGRAMED UNIT ON SUMMING NUMBER SERIES"

INTRODUCTORY PROGRAM

INSTRUCTIONS:

How would you like to be able to sum all the odd numbers from 1 to 51 without adding each number separately?

1 3 5 7 9 11 13 15 17 . . . 35 37 39 41 43 45 47 49 51 = 676

You are going to learn some short-cut ways for doing just that. You will be able to sum a series of numbers without adding each number in the series. For example, the short-cut for getting the sum of the above series is 26 x 26 = 676.

Answers	
1, 15,	1. The row of numbers below is called a number-series. In this particular series of numbers, each number is two more than the number preceding it in the series. Series: 1 3 5 7 9 11 13 ____ Each number in the series is called a term-value. What term-value beings the series? _____ What term-value comes after 13? _____
term-value 15	2. Series: 1 3 5 7 9 11 13 ____ The third term-value in the series is 5. The fourth ____-_____ in the series is 7. The eighth term-value in the series is? _____ NOTE: Hereafter when we are referring to a term-value in a series we shall sometimes use the capital letter "T" instead of writing out the word term-value.
5, 7, 9, 11, 13 T sub n, 5 3, 5	3. (T) Series: 1 3 5 7 9 ____ ____ T stands for term-value in the above series T = 1, 3, _, _, _, _, or _. n stands for the term-number or position of a term-value in the series counting from left to right. For example, when n = 4 (the fourth term-value in the series) T_n (pronounced T sub n) = 7; Likewise, when n = 3, T_n (pronounced _ ___ _) = _____. when n = 2, T_n = _____. when T_n = 9, n = _____.

Answers	
T sub n, 4	4. (n) Term No.: 1 2 3 $\boxed{4}$ 5 6 7
11,	(T) Series : 1 3 5 $\boxed{7}$ 9 _ _
7	When T_n (pronounced _ ___ _) = 7, n = _____.
	When n = 6, T_n = _____.
	When T_n = 13, n = _____.
5	5. (n) Term No.: 1 2 3 4 5 6 7
2	(T) Series : 1 3 5 7 9 _ _
	When n = 3, T_n = _____.
	When T_n = 3, n = _____.
3	6. (n) Term No.: 1 2 $\boxed{3}$ 4 $\boxed{5}$ 6 7
9	(T) Series : 1 3 $\boxed{5}$ 7 $\boxed{9}$ _ _
13	When T_n = 5, n = _____.
4	When n = 5, T_n = _____.
	When n = 7, T_n = _____.
	When T_n = 7, n = _____.
7, 5	7. (n) Term No.: 1 2 3 4 5 6 7
13	(T) Series : 1 3 5 7 9 _ _
	When n = 4, n + 1 = 5, n - 1 = 3.
	When n = 6, n + 1 = _, n - 1 = _.
	When n = 4, T_{n+1} (pronounced T sub n + 1) = 9.
	When n = 6, T_{n+1} = _____.
7	8. (n) Term No.: 1 2 3 4 5 6 7
11	(T) Series : 1 3 5 7 9 _ _
	The symbol $\neq$ stands for "is not dqual to"
	When T_n = 5, $T_{n+1} \neq 6$, rather T_{n+1} = _____.
	Likewise, when T_n = 9, $T_{n+1} \neq 10$, rather T_{n+1} = _____.

Answers	
11, 13, 9	9. (n) Term No.: 1 2 3 4 5 6 7 (T) Series : 1 3 5 7 9 _ _ When $n = 2$, $T_n = 3$, $T_{n+1} = 5$, $T_{n-1} = 1$. When $n = 6$, $T_n =$ _, $T_{n+1} =$ _, $T_{n-1} =$ _.
T sub n, 9 T sub n + 1, 11 T sub n - 1, 7 value number	10. (n) Term No.: 1 2 3 4 5 6 7 (T) Series : 1 3 5 7 9 _ _ Now let's review. When $n = 5$, T_n (pronounced _ ___ _) = ____ T_{n+1} (pronounced _ ___ _ _) = ____ T_{n-1} (pronounced _ ___ _ _) = ____ T is a symbol which stands for term-_____. n is a symbol which stands for term-_____.
No answer required. Go on to the next problem.	11. Now we are going to learn short-cut methods for summing the terms of a number series. You'll be able to look at a number series like the following: (n) Term No.: 1 2 3 4 5 6 7 8 9 10 11 12 13 1 3 5 7 9 11 13 15 17 19 21 23 25 14 15 16 17 18 19 20 21 27 29 31 33 35 37 39 41 and without adding be able immediately (in 5 or 10 seconds) to tell that the sum of all 21 numbers in the series is equal to 441. Go on to the next problem.
n 1, 3, 5, 7, 9, five	12. (n) Term No.: 1 2 3 4 5 6 7 (T) Series : 1 3 5 7 9 11 13 Σ means the sum of. If a series of n terms (n refers to whatever number of terms we are talking about) then we can talk about $\overset{n}{\Sigma}$ or the sum of ___ terms. In the above series if $n = 7$, then $\overset{n}{\Sigma}$ means the sum of 1, 3, 5, 7, 9, 11, 13, or the sum of the first 7 terms of the series. In the above series if $n = 5$, then $\overset{n}{\Sigma}$ means the sum of which terms in the number series? _, _, _, _, _, or the first terms of the series.

Answers	
three	13. (n) Term No.: 1 2 3 4 5 6 7
16, the first four terms of the series.	(T) Series : 1 3 5 7 9 _ _
	$\overset{n}{\Sigma}$ means the sum of all terms in the series from the first to the nth.
	In this series when n = 2, the sum of n terms or $\overset{n}{\Sigma}$ is equal to 4.
	When $n = 3$, $\overset{n}{\Sigma} = 9$ or the sum of the first ___ terms of the series.
	When $n = 4$, $\overset{n}{\Sigma} =$ ___ or the sum of ___ ___ ___ ___ ___ ___ ___.
25, the sum of the first five terms of the series.	14. (n) Term No.: 1 2 3 4 5 6 7
	(T) Series : 1 3 5 7 9 _ _
5,	When $n = 5$, $\overset{n}{\Sigma} =$ ____ or ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___
3	When $\overset{n}{\Sigma} = 25$, n = ____.
	When $\overset{n}{\Sigma} = 9$, n = ____.
2	15. (n) Term No.: 1 2 3 4 5
5, 9	(T) Series : 1 3 5 7 9
7, 16	If n = 3, n + 1 is 4 and n - 1 is? ____
5, 9	If n = 3, $T_n =$ ____, $\overset{n}{\Sigma} =$ ____ (The sum of the first n terms)
5	If n = 3, $T_{n+1} =$ ____, $\overset{n+1}{\Sigma} =$ ____ (The sum of the first n + 1 terms)
	If n = 4, $T_{n-1} =$ ____, $\overset{n-1}{\Sigma} =$ ____ (The sum of the first n - 1 terms)
	If $T_n = 9$, n = ____.

Answers		
the sum of the first five terms of the series.	16.	(n) Term No.: 1 2 3 4 5 (T) Series : 1 3 5 7 9
the sum of the first n + 1 or four terms of the series.		Write out in words the meaning of the following: When n = 5, $\overset{n}{\Sigma} = \underline{25}$ or . . . When n = 3, $\overset{n+1}{\Sigma} = \underline{16}$ or . . .
5, 9	17.	(n) Term No.: 1 2 3 4 5 6 7
9, 25		(T) Series : 1 3 5 7 9 _ _
11, 36		When n = 3, T_n = _, $\overset{n}{\Sigma}$ = ____.
11, 36		When n = 5, T_n = _, $\overset{n}{\Sigma}$ = ____.
7, 16		When n = 6, T_n = _, $\overset{n}{\Sigma}$ = ____.
5, 3		When n = 5, T_{n+1} = _, $\overset{n+1}{\Sigma}$ = ____.
5, 3, 2		When n = 5, T_{n-1} = _, $\overset{n-1}{\Sigma}$ = ____.
		When $\overset{n}{\Sigma}$ = 9 T_n = _, n = ____.
		When $\overset{n+1}{\Sigma}$ = 9, T_{n+1} = _, T_n = _, n = ____.
5, 9	18.	(n) Term No.: 1 2 3 4 5 6 7
3, 5		(T) Series : 1 3 5 7 9 _ _
5, 6		If $\overset{n+1}{\Sigma}$ = 36, then $\overset{n}{\Sigma}$ = 25, n = ____ T_n = ____.
2, 1		If $\overset{n+1}{\Sigma}$ = 16, n = ____, T_n = ____.
		If T_n = 9, n = ____, n + 1 = ____.
		If T_n = 3, n = ____, n - 1 + ____.
10, 6	19.	There are many different kinds of series. Take this one:
20, five		(n) Term No.: 1 2 3 4 5 6 7
		(T) Series : 0 2 4 6 8 _ _
		If T_n = 8, n = 5, T_{n+1} = ___ T_{n-1} = ____.
		If n = 5, $\overset{n}{\Sigma}$ = ____ or the sum of the first ____ terms.

Answers

6, 8, 4, 12

20. (n) Term No.: 1 2 3 4 5 6 7

(T) Series : 0 2 4 6 8 _ _

If n = 4, T_n ___, T_{n+1} = ___, T_{n-1} = ___, $\overset{n}{\Sigma}$ = ___.

$\overset{n}{\Sigma}$

21. (n) Term No.: 1 2 3 4 5

(T) Series : 0 2 4 6 8

When n = 4, $\overset{n}{\Sigma}$ is a symbol which means "the sum of the first four terms in the series."

When n = 3, ___ is a symbol which means "the sum of the first three terms in the series."

When n = 5, ___ is a symbol which means "______________."

$\overset{n}{\Sigma}$, the sum of the first five terms in the series.

$\overset{n}{\Sigma} = 6$

22. (n) Term No.: 1 2 3 4 5

(T) Series : 0 2 4 6 8

When n = 4, the expression $\overset{n}{\Sigma} = 12$ means "the sum of (Σ) the term values of the first four terms is equal to 12."

For n = 3, write the expression which means "the sum of the term values of the first three terms is equal to 6."

$\overset{n}{\Sigma} = 20$

23. (n) Term No.: 1 2 3 4 5

(T) Series : 0 2 4 6 8

For n = 5, write the expression which means "the sum of the term values of the first five terms is equal to 20."

$\overset{n+1}{\Sigma} = 30$

$\overset{n-1}{\Sigma} = 12$

24. (T) Series: 0 2 4 6 8 10

For n = 4, the expression $\overset{n+1}{\Sigma} = 20$ means "the sum of the term values of the first five terms is equal to 20."

For n = 4, the expression $\overset{n-1}{\Sigma} = 6$ means "the sum of the term values of the first three terms is equal to 6."

For n = 5, write the expression which means "the sum of the term values of the first six terms is equal to 20." ______________

For n = 5, write the expression which means "the sum of the term values of the first four terms is equal to 12." ______________

Answers		
5, 9	25.	(n) Term No.: 1 2 3 4 5 6 7
		(T) Series : 1 3 5 7 9 11 13
		(Σ) Sums : 1 4 9 16 25 36 49
		Instead of just writing down a number series and term numbers we can also write the cumulative sums (Σ's) below the series as shown above.
		When n = 4, T_n = 7, and $\overset{n}{\Sigma}$ = 16.
		When n = 3, T_n = _____, and $\overset{n}{\Sigma}$ = _____.
11, 36	26.	(n): 1 2 3 4 5 6 7
T_n		(T): 1 3 5 7 9 11 13
n		(Σ): 1 4 9 16 25 36 49
Σ		When n = 6, T_n = ___ and $\overset{n}{\Sigma}$ = ___.
$\overset{n}{\Sigma}$		The symbol for term-values is _____.
six		The symbol for term number or position of a term-value in the series is _____.
		The symbol meaning "the sum of" is _____.
		The symbol for "the sum of n terms in a series" is _____.
		When n = 5, $\overset{n+1}{\Sigma}$ means the sum of the first _____ terms in the series.
T_n	27.	The symbol for "term value" is _____.
Σ		The symbol for "the sum of" is _____.
n		The symbol for "term number" is _____.
$\overset{n}{\Sigma}$		The symbol for "the sum of the first n terms in a series is _____.
7 5		When n = 6, $\overset{n+1}{\Sigma}$ means the sum of the first _____ terms in the series.
		When n = 6, $\overset{n-1}{\Sigma}$ means the sum of the first _____ terms in the series.

Answers

without

28.

(n):	1	2	3	4	5
(T):	1	3	5	7	9
(Σ):	1	4	9	16	25

Now what we want you to do is to write some short-cut rules for finding the sum of the terms in a series without adding them separately.

Thus, in the above series there are 5 terms (n = 5) and we will teach you later how to sum these 5 term values _____ adding separately each number in the series.

16

29.

(n):	1	2	3	4	5
(T):	1	3	5	7	9
(Σ):	1	4	9	16	25

If n = 5, $\overset{n}{\Sigma} = 25$. That means that if n = 5, the term values 1 plus 3, plus 5, plus 7, plus 9 equal 25.

Likewise, if n = 4, $\overset{n}{\Sigma} =$ _____.

9

without

30.

(n):	1	2	3	4	5
(T):	1	3	5	7	9
(Σ):	1	4	9	16	25

If n = 3, $\overset{n}{\Sigma} =$ _____.

If n = 3, $\overset{n}{\Sigma}$ means the sum of the first three term values in the series or the sum of 1, 3, and 5.

However, we will learn to sum a number series by a short-cut procedure _____ adding each of the term values separately.

3, 5, 1

9, 5, 7, 3

25, 9, 11, 7

31.

(n):	1	2	3	4	5
(T):	1	3	5	7	9
(Σ):	1	4	9	16	25

When n = 2, $\overset{n}{\Sigma} = 4$, $T_n =$ ___, $T_{n+1} =$ ___, $T_{n-1} =$ ___.

When n = 3, $\overset{n}{\Sigma} =$, $T_n =$ ___, $T_{n+1} =$ ___, $T_{n-1} =$ ___.

When n = 5, $\overset{n}{\Sigma} =$, $T_n =$ ___, $T_{n+1} =$ ___, $T_{n-1} =$ ___.

Answers

9, 8

5, 6

T_n

$T_{n - 1}$

$T_{n + 1}$

32. (n): 1 2 3 4 5

(T): [illegible] 3 5 7 9

(Σ): 1 4 9 16 25

When n = 4, $T_{n + 1}$ = ___ but $T_n + 1$ = _____.

When n = 4, $T_{n - 1}$ = ___ but $T_n - 1$ = _____.

For n = 3, what is the symbol that identifies "5" in the box? _____

For n = 4, what is the symbol that identifies "5" in the box? _____

For n = 2, what is the symbol that identifies "5" in the box? _____

7, 6

33. (n): 1 2 3 4 5 6

(T): 1 3 5 7 9

(Σ): 1 4 9 16 25

When n = 2, $T_{n + 1} = 5$ and $T_n + 1 = 4$

Do you see that $T_{n + 1}$ means: the value of the term "n = 1"

and $T_n = 1$ means: add one to the value of term "n"

When n = 3, $T_{n + 1}$ = _____ and $T_n + 1$ = _____

Term "n + 1"

The value of term "n"

11, 10

16, 7, 5

34. (n): 1 2 3 4 5 6

(T): 1 3 5 7 9

(Σ): 1 4 9 16 25

$T_{n + 1}$ means: the value of __________

$T_n + 1$ means: add one to ____________

When n = 5, $T_{n + 1}$ __________, $T_n + 1$ _______

For n = 4, $\overset{n}{\Sigma}$ = __________, T_n = __________, $T_{n - 1}$ = __________

EXPOSITORY TREATMENT

Answers

4, 8

16, 5

E35.

(n) Term No.:	1	2	3	4	5	6	7	8	9
(T) Series :	1	2	4	8	16	32	64		
				↑↗	↑↗				
(Σ) :	1	3	7	15	31	63			

In the following items you will occasionally see diagonal or vertical arrows like those above.

These arrows give hints as to how you can get $\overset{n}{\Sigma}$ from either n to T_n or both N and T_n.

For example, the vertical arrow pointing up from 31 means that in this series when n = 5, $\overset{n}{\Sigma}$ (the sum "31") can be obtained from either n (in this case 5) or T_n (in this case 16) or both. Likewise the sum "15" (following the vertical arrow) can be obtained from n (in this case ___) or T_n (in this case ___) or both.

Also, the diagonal arrow from 31 to the next column means that the sum "31" can be obtained from T_{n+1} (in this case 32) or n + 1 (in this case 6) or both. Likewise the sum "15" (following the diagonal arrow) can be obtained from T_{n+1} (in this case ___) or n + 1 (in this case ___) or both.

$T_{n+1} - 1$

The sum of the first n terms in the series is equal to one less than the term value of the $n + 1^{th}$ term in the series.

$\overset{n}{\Sigma} = 64 - 1$

$\overset{n}{\Sigma} = 63$

E36.

(n) Term No.:	1	2	3	4	5	6	7	8
(T) Series :	1	2	4	8	16	32	64	
	↗	↗	↗	↗	↗			
(Σ) :	1	3	7	15	31			

Here is the same series with connecting arrows to clarify the rule for summing the series.

Rule: The rule for getting the sum of this series is: in symbols: $\overset{n}{\Sigma} = T_{n+1} - 1$; in words: the sum of the first n terms in the series is equal to one less than the term value of the $n + 1^{th}$ term in the series.

Example: This is how the rule works. Suppose we want the sum of the first 5 terms in the series. Then n = 5 and $T_{n+1} = 32$. Thus, for n = 5, in this series, $\overset{n}{\Sigma} = 32 - 1$ or simplifying the answer $\overset{n}{\Sigma} = 31$.

Rule: Copy the rule: in symbols: $\overset{n}{\Sigma} =$

in words: "the sum of _______

Example: Use the formula for working out your answer to this problem. When n = 6, $\overset{n}{\Sigma} =$ _____ or simplifying the answer $\overset{n}{\Sigma} =$ _____

Answers

The sum of the first n terms in the series is equal to one less than the term value of the $n + 1^{th}$ term.

31

E37. (n) Term No.: 1 2 3 4 5 6 7 8

(T) Series : 1 2 4 8 16 32 64

(Σ) : 1 3 7 15 31 ___ ___

When n = 2, $\overset{n}{\Sigma} = 3$

When n = 3, $\overset{n}{\Sigma} = 7$

When n = 4, $\overset{n}{\Sigma} = 15$

Complete the rule: $\overset{n}{\Sigma}$ (the sum of the first n terms in this series) is equal to ____________

Use the rule for figuring the answer to this problem.

When n = 5, $\overset{n}{\Sigma} =$ ____________

$T_{n+1} - 1$

The sum of the first n terms in the series is equal to one less than the value of the $n + 1^{th}$ term.

127

E38. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32 64

(Σ): 1 3 7 15 31 ___ ___

When n = 5, $\overset{n}{\Sigma} = 31$

When n = 6, $\overset{n}{\Sigma} = 63$

When n = 7, $\overset{n}{\Sigma} =$ ___

State the rule for summing the number series in symbols: $\overset{n}{\Sigma} =$ ____________

in words: "The sum of the first n terms in the series is equal to ____________

15

1023

255

E39. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32 64 ___

(Σ): 1 3 7 15 31 ___ ___ ___

You have stated the rule for this series that $\overset{n}{\Sigma} = T_{n+1} - 1$

When $T_{n+1} = 16$, $\overset{n}{\Sigma} =$ ________

When $T_{n+1} = 1024$, $\overset{n}{\Sigma} =$ ________

When $T_n = 128$, $\overset{n}{\Sigma} =$ ________ (This one is tricky.)

Answers

$\overset{n}{\Sigma} = T_{n+1} - 1$

The sum of the first n terms in the series is equal to one less than the term value of the $n + 1^{th}$ term in the series. (NOTE: The words need not be exactly as above as long as the meaning is correct.)

127

255

511

511

2047

E40.

(n):	1	2	3	4	5	6	7	8	9
(T):	1	2	4	8	16	32	64	128	256
(Σ):	1	3	7	15	31	63			

Rule: The rule for getting the sum of this series is

in symbols: $\overset{n}{\Sigma} =$

in words: the sum of the first n terms in the series is equal to __________

Examples: Use the rule for working out your answers.

When n = 7, $\overset{n}{\Sigma} =$ __________

When n = 8, $\overset{n}{\Sigma} =$ __________

When n = 9, $\overset{n}{\Sigma} =$ __________

When $T_n = 256$, $\overset{n}{\Sigma} =$ __________

When $T_{n+1} = 2048$, $\overset{n}{\Sigma} =$ __________

13

15

E41. Now we can go on to another series. Remember, we are always looking for a rule for getting $\overset{n}{\Sigma}$ by using the term number (n) or the term-value (T_n) or both.

(n):	1	2	3	4	5	6	7	8
(T):	1	3	5	7	9	11	__	__

What is T_n when n = 7? _____

What is T_n when n = 8? _____

Answers

4^2, 16

$\overset{n}{\Sigma} = n^2$

The sum of the first n terms in the series is equal to the square of the term number.

n^2

The sum of the first *n terms in* the series is equal to the square of the term number.

16

25

E42. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11 __ __

(Σ): 1 4 9

Rule: One rule for summing this series is

in symbols: $\overset{n}{\Sigma} = n^2$

in words: The sum of the first n terms in the series is equal to the square of the term number.

Example: This is how the rule works.

For n = 2, $\overset{n}{\Sigma} = 2^2$ or $\overset{n}{\Sigma} = 4$

For n = 3, $\overset{n}{\Sigma} = 3^2$ or $\overset{n}{\Sigma} = 9$

For n = 4, $\overset{n}{\Sigma} =$ ____ or $\overset{n}{\Sigma} =$ ____

Rule: Copy the rule:

in symbols: $\overset{n}{\Sigma} =$ ____

in words: The sum of the first n terms in the series is equal to ____________________

E43. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11 __ __

(Σ): 1 4 9

Rule: The rule for getting the sum of this series is:

in symbols: $\overset{n}{\Sigma} =$ ____

in words: The sum of the first n terms in the series is equal to ____________________

Examples: Use the rule for working out your answer.

When n = 4, $\overset{n}{\Sigma} =$ ____

When n = 5, $\overset{n}{\Sigma} =$ ____

Answers

4, 7, 9, 5

5, 9, 11, 7

E44.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & \_\_ & \_\_ \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & & & \end{array}$$

For $\overset{n}{\Sigma} = 16$, $n = \_\_$, $T_n = \_\_$, $T_{n+1} = \_\_$, $T_{n-1} = \_\_$

For $\overset{n}{\Sigma} = 25$, $n = \_\_$, $T_n = \_\_$, $T_{n+1} = \_\_$, $T_{n-1} = \_\_$

As in the other series we are always learning a rule for getting $\overset{n}{\Sigma}$ by using either n or T_n. In other words by adding to, subtracting from, multiplying or dividing the n, or T_n, or T_{n+1}, or T_{n-1}, we will be able to get $\overset{n}{\Sigma}$.

(The arrow gives the hint that you can get $\overset{n}{\Sigma}$ from either T_n or n.)

16

$\overset{n}{\Sigma} = n^2$

E45.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & \_\_ & \_\_ \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & 36 & & \end{array}$$

When T_n is 3, $\overset{n}{\Sigma}$ is 4, or $T_n + 1$

When T_n is 5, $\overset{n}{\Sigma}$ is 9, or $T_n + 4$

When T_n is 7, $\overset{n}{\Sigma}$ is 16, or $T_n + \_\_$

There is no simple rule for getting $\overset{n}{\Sigma}$ from T_n.

When n is 2, $\overset{n}{\Sigma}$ is 4

When n is 3, $\overset{n}{\Sigma}$ is 9

When n is 4, $\overset{n}{\Sigma}$ is __

We showed you a simple rule for getting $\overset{n}{\Sigma}$ from n. The rule is: $\overset{n}{\Sigma} = \_\_\_\_\_\_\_\_$

2

3

E46.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & & \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & 36 & & \end{array}$$

There is a simple relationship between $\overset{n}{\Sigma}$ and n.

When n is 2, $\overset{n}{\Sigma}$ can be obtained by multiplying n by _____.

When n is 3, $\overset{n}{\Sigma}$ can be obtained by multiplying n by _____.

Answers

5 x 5 or 25

Yes

E47. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11
↑ ↑ ↑ ↑ ↑ ↑
(Σ): 1 4 9 16 25 36

Another way of saying it is $\overset{n}{\Sigma}$ can be gotten by multiplying n by itself. This means, when n is 5, $\overset{n}{\Sigma}$ should be _____.

Does this check out?

n^2

The sum of the first n term in the series is equal to the square of the term number.

E48. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11
↑ ↑ ↑ ↑ ↑ ↑
(Σ): 1 4 9 16 25 36

Now you state the general rule for summing this series

in symbols: $\overset{n}{\Sigma}$ = __________

in words: The sum of the first n terms in this series is equal to __________

64

E49. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11
↑ ↑ ↑ ↑ ↑ ↑
(Σ): 1 4 9 16 25 36

The rule is $\overset{n}{\Sigma} = n \times n$, or n^2. Are you confident about this?

What is $\overset{n}{\Sigma}$ when n = 8? _____ Check this out; use the back of the previous card for scratch paper if you need to.

81

121

144

100

E50. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11
↑ ↑ ↑ ↑ ↑ ↑
(Σ): 1 4 9 16 25 36

When n = 9, $\overset{n}{\Sigma}$ = _____

When n = 11, $\overset{n}{\Sigma}$ = _____

When n = 12, $\overset{n}{\Sigma}$ = _____

The sum of this series through term-value nineteen (T = 19) is ________.

Answers

127

E51. Now that you're getting pretty good at series, let's try another one.

(n):	1	2	3	4	5	6	7	8	9
(T):	1	2	4	8	16	32	64	128	___
	↑	↑	↑	↑	↑	↑			
(Σ):	1	3	7	15	31	63	___	___	___

Actually, this is the same one we looked at the time before last. But notice we are looking for a different relationship this time. Notice how the arrows point this time. Fill in the sum for the bottom row, under n = 7.

(2 x 16) - 1, 31

$\overset{n}{\Sigma} = 2T_n - 1$

The sum of the first n terms in the series is equal to two times the term value of the n^{th} term, minus one or one less than two times the value of the n^{th} term.

E52.

(n):	1	2	3	4	5	6	7	8
(T):	1	2	4	8	16	32	___	___
(Σ):	1	3	7	15	___	___	___	___

Rule: We had the rule $\overset{n}{\Sigma} = T_{n+1} - 1$ for summing this series before. Another rule for getting the sum of this series is:

in symbols: $\overset{n}{\Sigma} = 2T_n - 1$

in words: The sum of the first n terms in the series is equal to 2 times the term value, minus one. Or in other words, $\overset{n}{\Sigma}$ = one less than 2 times the term value.

Examples: This is how the rule works:

For n = 3, $\overset{n}{\Sigma} = (2 \times 4) - 1$ or $\overset{n}{\Sigma} = 7$

For n = 4, $\overset{n}{\Sigma} = (2 \times 8) - 1$ or $\overset{n}{\Sigma} = 15$

For n = 5, $\overset{n}{\Sigma} = (\quad)$ ___ or $\overset{n}{\Sigma} =$ ___

Rule: Copy the rule here.

in symbols: $\overset{n}{\Sigma} =$ ______________________________

in words: The sum of the first n terms in the series is equal to ______________________________

Answers

$2T_n - 1$

The sum of the first n terms in this series is equal to two times the term value, minus one.

or

one less than two times the term value.

E53. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32

(Σ): 1 3 7 15

A rule for summing this series using T_n is:

in symbols: $\overset{n}{\Sigma}$ = ________

in words: The sum of the first n terms in the series is equal to ______________

Examples: (Use the formula)

31 — For n = 5, $\overset{n}{\Sigma}$ = ____

63 — For n = 6, $\overset{n}{\Sigma}$ = ____

127 — For n = 7, $\overset{n}{\Sigma}$ = ____

8

E54. (n): 1 2 3 4 5 6 7 8 9

(T): 1 2 4 8 16 32 64

↑ ↑ ↑ ↑ ↑ ↑

(Σ): 1 3 7 15 31 63

This time we have a relation between $\overset{n}{\Sigma}$ and T_n.

When $\overset{n}{\Sigma} = 3$, $T_n = 2$

When $\overset{n}{\Sigma} = 7$, $T_n = 4$

When $\overset{n}{\Sigma} = 15$, T_n = ____

2

E55. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32 64 __

↑ ↑ ↑ ↑ ↑ ↑

(Σ): 1 3 7 15 31 63 __ __

You can almost get $\overset{n}{\Sigma}$ by multiplying T_n by ________________.

subtracting one

E56. (n): 1 2 3 4 5 6 7 8 9

(T): 1 2 4 8 16 32 64 __ __

↑ ↑ ↑ ↑ ↑ ↑

(Σ): 1 3 7 15 31 63 __ __ __

When $\overset{n}{\Sigma} = 3$, $T_n = 2$ Adding 1 to T_n certainly doesn't always work to get $\overset{n}{\Sigma}$.

When $\overset{n}{\Sigma} = 7$, $T_n = 4$ Dividing won't work.

When $\overset{n}{\Sigma} = 15$, $T_n = 8$ We showed, however, that you can get $\overset{n}{\Sigma}$ from T_n by multiplying T_n by 2 and ____.

Answers

2, 4, 8, 16, 32, 64

subtracting one

E57.

$$\begin{array}{l|ccccccccc}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 & 64 & \underline{\quad} & \underline{\quad} \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \\
(\Sigma): & 1 & 3 & 7 & 15 & 31 & 63 & \underline{\quad} & \underline{\quad} & \underline{\quad} \\
T_n \times 2 & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad}
\end{array}$$

Write out the fourth row.

In each case you can get $\overset{n}{\Sigma}$ by taking T_n x 2 and ____________________.

31

E58.

$$\begin{array}{l|cccccccc}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 & 64 & \underline{\quad} \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\
(\Sigma): & 1 & 3 & 7 & 15 & 31 & 63 & \underline{\quad} & \underline{\quad}
\end{array}$$

When T_n is 8, T_n x 2 = 16, $\overset{n}{\Sigma}$ = 15.

When T_n is 16, T_n x 2 = 32, $\overset{n}{\Sigma}$ = _____.

$2T_n - 1$

$T_{n+1} - 1$

E59.

$$\begin{array}{l|cccccccc}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 & & \\
 & & & \nearrow & \uparrow & & & & \\
(\Sigma): & 1 & 3 & 7 & 15 & & & &
\end{array}$$

This series may be summed with at least two rules: One that uses T_n and one that uses T_{n+1}. Write both of the rules in symbols.

$\overset{n}{\Sigma}$ = __________ (A rule using T_n)

$\overset{n}{\Sigma}$ = __________ (A rule using T_{n+1})

one

127

E60. Now here is a slightly more complex series. First let's just look at the series itself.

Series: 1 3 7 15 31 63 __ __

It's almost as if each term were multiplied by two, but that isn't quite right. Actually, each term is the preceding term multiplied by two, plus _______. Check this out. What is the term after 63? _________

Answers

127 - (6 + 1), 120

$\overset{n}{\Sigma} = T_{n+1} - (n + 1)$

The sum of the first n terms in the series, is equal to the term value of the n + 1th term, minus n + 1.

E61.

(n):	1	2	3	4	5	6	7	8
(T):	1	3	7	15	31	63	___	___
(Σ):	1	4	11	26	57	___	___	___

Problem: Find a rule for the sum of this series using term number (n) or term value (T_n) or both.

Rule: One rule is:

in symbols: $\overset{n}{\Sigma} = T_{n+1} - (n + 1)$;

in words: the sum of the first n terms in the series is equal to the term value of the n + 1th term, minus n + 1.

Examples: This is how the rule works:

For n = 3, $\overset{n}{\Sigma} = 15, - (3 + 1)$ or $\overset{n}{\Sigma} = 11$

For n = 5, $\overset{n}{\Sigma} = 63 - (5 + 1)$ or $\overset{n}{\Sigma} = 57$

For n = 6, $\overset{n}{\Sigma}$ = ___________ or $\overset{n}{\Sigma}$ = ___

Copy the rule here in symbols: $\overset{n}{\Sigma}$ = ___________

in words: The sum of the first n terms in the series is equal to ___________________________.

$T_{n+1} - (n + 1)$

The sum of the first n terms in the series is equal to the term value of the n + 1th term, minus n + 1.

or

subtract n + 1 from the term value of the n + 1th term.

63 - (5 + 1), 57

127 - (6 + 1), 120

254 - (7 + 1), 246

120

E62.

(n):	1	2	3	4	5	6	7	8
(T):	1	3	7	15	31	63	127	___
(Σ):	1	4	11	26	57	___	___	___

A rule for summing this series is:

in symbols: $\overset{n}{\Sigma}$ = _________

in words: The sum of the first n terms in the series is equal to ___________________________.

Examples: (Use the rule)

For n = 5, $\overset{n}{\Sigma}$ = _____

For n = 6, $\overset{n}{\Sigma}$ = _____

For n = 7, $\overset{n}{\Sigma}$ = _____

When $T_{n+1} = 127$, $\overset{n}{\Sigma}$ = _____

Answers

and (or minus)

E63.

$$\begin{tikzcd}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_ & \_
\end{tikzcd}$$

Look at the arrows. They may remind you that the formula you just learned for summing this series gets $\overset{n}{\Sigma}$ from T_{n+1} _______ $n + 1$.

31

E64.

$$\begin{tikzcd}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_ & \_
\end{tikzcd}$$

When $\overset{n}{\Sigma} = 4$, $T_{n+1} = 7$

When $\overset{n}{\Sigma} = 11$, $T_{n+1} = 15$

When $\overset{n}{\Sigma} = 26$, $T_{n+1} =$ ______

- (minus) 4

-5

E65.

$$\begin{tikzcd}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_ & \_
\end{tikzcd}$$

When $\overset{n}{\Sigma} = 4$, $T_{n+1} = 7$	We cannot get from T_{n+1} to $\overset{n}{\Sigma}$ by addition.
When $\overset{n}{\Sigma} = 11$, $T_{n+1} = 15$	We can get from T_{n+1} to $\overset{n}{\Sigma}$ by subtraction.
When $\overset{n}{\Sigma} = 26$, $T_{n+1} = 31$	When $T_{n+1} = 7$, $\overset{n}{\Sigma} = 4$, or $T_{n+1} - 3$
	When $T_{n+1} = 15$, $\overset{n}{\Sigma} = 11$, or T_{n+1} ______
	When $T_{n+1} = 31$, $\overset{n}{\Sigma} = 26$, or T_{n+1} ______

Answers

6

E66.

$$\begin{tikzcd}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_ & \_
\end{tikzcd}$$

When we have $T_{n+1} = 7$, we can get $\overset{n}{\Sigma}$ by subtracting 3,

When we have $T_{n+1} = 15$, we can get $\overset{n}{\Sigma}$ by subtracting 4.

When we have $T_{n+1} = 31$, we can get $\overset{n}{\Sigma}$ by subtracting 5.

When we have $T_{n+1} = 63$, we can get $\overset{n}{\Sigma}$ by subtracting __.

6

E67.

$$\begin{tikzcd}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_ & \_
\end{tikzcd}$$

When $T_{n+1} = 7$, then $\overset{n}{\Sigma} = 7 - 3$

When $T_{n+1} = 15$, then $\overset{n}{\Sigma} = 15 - 4$

When $T_{n+1} = 31$, then $\overset{n}{\Sigma} = 31 - 5$

(NOTE: The number we subtract from T_{n+1} is the number directly above (i.e., n + 1).)

For example, when $T_{n+1} = 63$ what do we subtract? _______

We hope you see it.

E68.

$$\begin{tikzcd}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_ & \_
\end{tikzcd}$$

When $T_{n+1} = 7$, then $\overset{n}{\Sigma} = 7 - 3$. When $T_{n+1} = 15$, then $\overset{n}{\Sigma} = 15 - 4$, etc.

What we are subtracting in each case is n + 1, isn't it? Look at T_{n+1} and then look at the number in the row above it (that is, at n + 1). Do you see that subtracting n + 1 from T_{n+1} you can $\overset{n}{\Sigma}$?

Answers	
	E69. Now let's review.
$T_{n+1} - 1$	(n): 1 2 3 4 5
$2T_n - 1$	(T): 1 2 4 8 _
31	(Σ): 1 3 7 15 __
16	Write two different rules in symbols.
	$\overset{n}{\Sigma}$ = ________
	$\overset{n}{\Sigma}$ = ________
	Complete the example.
	for n = 5, $\overset{n}{\Sigma}$ = ________
	for n = 5, T_n = ________
n^2 or n x n	(n): 1 2 3 4 5
25	(T): 1 3 5 7 _
9	(Σ): 1 4 9 16 __
	Write the rule in symbols.
	$\overset{n}{\Sigma}$ = ________
	Complete the example.
	for n = 5, $\overset{n}{\Sigma}$ = ________
	for n = 5, T_n = ________
$T_{n+1} - (n + 1)$	(n): 1 2 3 4 5
57	(T): 1 3 7 15 __
31	(Σ): 1 4 11 26 __
	Write the rule in symbols.
You have completed the program! Congratulations!	$\overset{n}{\Sigma}$ = ________
	Complete the example.
	for n = 5, $\overset{n}{\Sigma}$ = ________
	for n = 5, T_n = ________

DISCOVERY TREATMENT

Answers

4, 8

16, 5

D35.

$$\begin{tikzcd}
\text{Term No. (n):} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{Series (T):} & 1 & 2 & 4 & 8 & 16 & 32 & 64 & & \\
(\Sigma): & 1 & 3 & 7 & 15 \arrow[u] \arrow[ur] & 31 \arrow[u] \arrow[ur] & 63 & & &
\end{tikzcd}$$

In the following items you will occasionally see diagonal or vertical arrows like those above.

These arrows give hints as to how you can get $\overset{n}{\Sigma}$ from either n or T_n or both n and T_n.

For example, the vertical arrow pointing up from 31 means that in this series when n = 5, $\overset{n}{\Sigma}$ (the sum "31") can be obtained from either n (in this case 5) or T_n (in this case 16) or both. Likewise the sum "15" (following the vertical arrow) can be obtained from n (in this case ___) or T_n (in this case ___) or both.

Also, the diagonal arrow from 31 to the next column means that the sum "31" can be obtained from T_{n+1} (in this case 32) or n + 1 (in this case 6) or both. Likewise the sum "15" (following the diagonal arrow) can be obtained from T_{n+1} (in this case ___) or n + 1 (in this case ___) or both.

Answers

Subtract one

Subtract one

Subtract one

The sum of the first n terms in the series is equal to the term value of the n + 1th term minus one.

31

D36.

$$\begin{tikzcd}
\text{Term No. (n):} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\text{Series (T):} & 1 & 2 & 4 & 8 & 16 & 32 & 64 & \_\_ \\
(\Sigma): & 1 \arrow[ur] & 3 \arrow[ur] & 7 \arrow[ur] & 15 \arrow[ur] & 31 \arrow[ur] & \_\_ & \_\_ & \_\_
\end{tikzcd}$$

Here is the same series with connecting arrows to give you a hint as to the rule for summing the series.

When n = 2, $\overset{n}{\Sigma} = 3$ (Think: What do you do to get 3 from the number the arrow points to?)

When n = 3, $\overset{n}{\Sigma} = 7$ (Think: What do you do to 8 to get 7?)

When n = 4, $\overset{n}{\Sigma} = 15$ (Think: What do you do to 16 to get 15?)

Complete the following rule for finding the sum of any number of terms in this series.

$\overset{n}{\Sigma}$ (the sum of the first n terms in this series) is equal to __________. Use the rule for figuring the answer to this problem.

When n = 5, $\overset{n}{\Sigma}$ = __________

127

$T_{n+1} - 1$

The sum of the first n terms in the series is equal to the term value of the n + 1th term minus 1.

or

one less than the value of the n + 1th term.

D37.

$$\begin{tikzcd}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 & 64 & \_\_ \\
(\Sigma): & 1 \arrow[ur] & 3 \arrow[ur] & 7 \arrow[ur] & 15 \arrow[ur] & 31 \arrow[ur] & \_\_ & \_\_ & \_\_
\end{tikzcd}$$

When n = 5, $\overset{n}{\Sigma} = 31$

When n = 6, $\overset{n}{\Sigma} = 63$

When n = 7, $\overset{n}{\Sigma}$ = __

Are you ready to state a general rule using symbols? Let T_{n+1} mean the term value of the n + 1th term and write a rule in symbols for summing n terms in this number series.

$\overset{n}{\Sigma}$ = __________

Now try stating the rule again in your own words. Complete the rule. "The sum of the first n terms in the series is equal to ______ ____."

Answers

15

1023

255

D38. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32 64 ___

(Σ): 1 ↗ 3 ↗ 7 ↗ 15 ↗ 31 ↗ ___ ___ ___

You have stated the rule for this series that $\overset{n}{\Sigma} = T_{n+1} - 1$

When $T_{n+1} = 16$, $\overset{n}{\Sigma} =$ _____

When $T_{n+1} = 1024$, $\overset{n}{\Sigma} =$ _____

When $T_n = 128$, $\overset{n}{\Sigma} =$ _____ (This one is tricky)

The sum of the first n terms in the series is equal to one less than the n + 1th term of the series.

$T_{n+1} - 1$

63

D39. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32 64 ___

(Σ): 1 ↗ 3 ↗ 7 ↗ 15 ↗ 31 ↗ ___ ___ ___

Rule: in symbols: $\overset{n}{\Sigma} = T_{n+1} - 1$

in words: The sum of the first n terms in the series is equal to one less than the term value of the n + 1th term.

Copy the rule here:

in words: ______________________________

in symbols: ______________________________

When n = 6, $\overset{n}{\Sigma} =$ __________

$T_{n+1} - 1$

The sum of the first n terms in the series is equal to one less than the value of the n + 1th term.

127

255

511

511

2047

D40. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32 64 ___

(Σ): 1 ↗ 3 ↗ 7 ↗ 15 ↗ 31 ↗ ___ ___ ___

Rule: The rule for getting the sum of this series is:

in symbols: $\overset{n}{\Sigma} =$ __________

in words: The sum of the first n terms in the series is equal to ______________________________

Examples:

When n = 7, $\overset{n}{\Sigma} =$ _____

When n = 8, $\overset{n}{\Sigma} =$ _____

When n = 9, $\overset{n}{\Sigma} =$ _____

When $T_n = 256$, $\overset{n}{\Sigma} =$ _____

When $T_{n+1} = 2048$, $\overset{n}{\Sigma} =$ _____

Answers

13

15

D41. Now we can go on to another series. Remember we are always looking for a rule for getting $\overset{n}{\Sigma}$ by using the term number (n), or the term value (T_n), or both.

$$\begin{array}{lcccccccc} (N): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & \_\_ & \_\_ \end{array}$$

What is T_n when n = 7? _____

What is T_n when n = 8? _____

4, 7, 9, 5

5, 9, 11, 7

D42.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & & \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & & & \end{array}$$

When $\overset{n}{\Sigma}$ = 16, n = ___, T_n = ___, T_{n+1} = ___, T_{n-1} = ___

When $\overset{n}{\Sigma}$ = 25, n = ___, T_n = ___, T_{n+1} = ___, T_{n-1} = ___

We will look for a rule for getting $\overset{n}{\Sigma}$ by using either n or T_n. In other words by adding to, subtracting from, multiplying, or dividing the n or T_n or T_{n-1} or T_{n+1} we will be able to get $\overset{n}{\Sigma}$.

(The arrow gives the hint that you can get $\overset{n}{\Sigma}$ from either n or T_n.)

9

No

16

There is a simple one. Go on to the next card.

D43.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & \_\_ & \_\_ \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & 36 & \_\_ & \_\_ \end{array}$$

When T_n is 3, $\overset{n}{\Sigma}$ is 4, or $T_n + 1$

When T_n is 5, $\overset{n}{\Sigma}$ is 9, or $T_n + 4$

When T_n is 7, $\overset{n}{\Sigma}$ is 16, or T_n + __

Is there a single rule for getting $\overset{n}{\Sigma}$ is you know T_n?

When n is 2, $\overset{n}{\Sigma}$ is 4

When n is 3, $\overset{n}{\Sigma}$ is 9

When n is 4, $\overset{n}{\Sigma}$ is __

Is there a simple rule for getting $\overset{n}{\Sigma}$ if you know n?

__Answers__

2

3

D44.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & \_\_ & \_\_ \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & 36 & \_\_ & \_\_ \end{array}$$

Do you see any relationship between $\overset{n}{\Sigma}$ and n?

When n is 2, $\overset{n}{\Sigma}$ can be obtained by multiplying n by _____.

When n is 3, $\overset{n}{\Sigma}$ can be obtained by multiplying n by _____.

5 x 5 or 25

Yes

D45.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & \_\_ & \_\_ \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & 36 & \_\_ & \_\_ \end{array}$$

It looks as though $\overset{n}{\Sigma}$ can be ogtten by multiplying n by itself.

This means when n is 5, $\overset{n}{\Sigma}$ should be _____. Does this check out?

n x n or n^2

The sum of the first __n__ terms in the series is equal to n times itself.

D46.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & \_\_ & \_\_ \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & 36 & \_\_ & \_\_ \end{array}$$

Now you state a general rule for summing this series.

in symbols: $\overset{n}{\Sigma}$ = ______________________________

in words: The sum of the first __n__ terms in this series is equal to __________________________

64

D47.

$$\begin{array}{lcccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (T): & 1 & 3 & 5 & 7 & 9 & 11 & \_\_ & \_\_ \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ (\Sigma): & 1 & 4 & 9 & 16 & 25 & 36 & \_\_ & \_\_ \end{array}$$

The rule is $\overset{n}{\Sigma} = n \times n$ or n^2. Are you confident about this?

What is $\overset{n}{\Sigma}$, when n = 8? _______ Check this out. Use your answer sheet for scratch paper if you need to.

Answers

n x n or n^2

The sum of the first _n_ terms in the series is equal to n times itself

or

the square of the term number.

16

25

16

n^2

The sum of the first _n_ terms in the series is equal to the square of the term number.

D48. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11 __ __

(Σ): 1 4 9

Rule: The rule for getting the sum of this series is:

in symbols: $\overset{n}{\Sigma}$ = ____________________

in words: The sum of the first _n_ terms in the series is equal to ____________________

Examples: Use the rule for working out your answer.

When n = 4, $\overset{n}{\Sigma}$ = _____

When n = 5, $\overset{n}{\Sigma}$ = _____

D49. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11 __ __

(Σ): 1 4 9

Rule: One rule for summing this series is:

in symbols: $\overset{n}{\Sigma} = n^2$

in words: The sum of the first _n_ terms in the series is equal to the square of the term number.

Example: This is how the rule works.

When n = 2, $\overset{n}{\Sigma} = 2^2$ or 2 x 2 or 4

When n = 3, $\overset{n}{\Sigma} = 3^2$ or 9

When n = 4, $\overset{n}{\Sigma}$ = _____

Rule: Copy the rule here.

in symbols: $\overset{n}{\Sigma}$ = ____________________

in words: The sum of the first _n_ terms in the series is equal to ____________________

Answers

81
121
144
100

D50. (n): 1 2 3 4 5 6 7 8

(T): 1 3 5 7 9 11

(Σ): 1 4 9 16 25 36

When n = 9, $\overset{n}{\Sigma}$ = ____

When n = 11, $\overset{n}{\Sigma}$ = ____

When n = 12, $\overset{n}{\Sigma}$ = ____

The sum of this series through term value nineteen (T_n = 19) is ____.

127

D51. Now that you're getting pretty good at series, let's try another one.

(n): 1 2 3 4 5 6 7 8 9

(T): 1 2 4 8 16 32 64 128 ___

(Σ): 1 3 7 15 31 63 ___ ___ ___

Actually, this is the same one we looked at the time before last. But notice we are looking for a different relationship this time. Notice how the arrows point this time. Fill in the sum for the bottom row, under n = 7.

8

D52. (n): 1 2 3 4 5 6 7 8 9

(T): 1 2 4 8 16 32 64 ___ ___

(Σ): 1 3 7 15 31 63 ___ ___ ___

Remember, we're again looking for a relation between $\overset{n}{\Sigma}$ and either T_n or n. Let's try T_n.

When $\overset{n}{\Sigma}$ = 3, T_n = 2

When $\overset{n}{\Sigma}$ = 7, T_n = 4

When $\overset{n}{\Sigma}$ = 15, T_n = __

Answers

multiplying

D53.

$$\begin{array}{lccccccccc}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 & 64 & \_\_ & \_\_ \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \\
(\Sigma): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ & \_\_ & \_\_
\end{array}$$

When $\overset{n}{\Sigma} = 3$, $T_n = 2$

When $\overset{n}{\Sigma} = 7$, $T_n = 4$

When $\overset{n}{\Sigma} = 15$, $T_n = 8$

Adding 1 to T_n certainly doesn't always work to get $\overset{n}{\Sigma}$. Dividing won't work. What might work in getting $\overset{n}{\Sigma}$ from T_n?

2

D54.

$$\begin{array}{lcccccccc}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 & 64 & \_\_ \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\
(\Sigma): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ & \_\_
\end{array}$$

It looks as though you could almost get $\overset{n}{\Sigma}$ by multiplying T_n by what? _____

Series: 1 2 4 8 16 32 64 __ __

(Σ): 1 3 7 15 31 63

T x 2: 1 4 8 16 32 64 128

D55.

$$\begin{array}{lccccccccc}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 & 64 & \_\_ & \_\_ \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \\
(\Sigma): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ & \_\_ & \_\_ \\
T_n \times 2: & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_
\end{array}$$

Write out the fourth row.

subtracting 1

In each case, you can get $\overset{n}{\Sigma}$ by taking T_n x 2 and _____ _____.

31

D56.

$$\begin{array}{lcccccccc}
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 & 64 & \_\_ \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\
(\Sigma): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ & \_\_
\end{array}$$

When T_n is 8, T_n x 2 = 16, $\overset{n}{\Sigma} = 15$.

When T_n is 16, T_n x 2 = 32, $\overset{n}{\Sigma} =$ _____

Answers

31

63

127

$2T_n - 1$

The sum of the first n terms is equal to two times the term value minus one

or

one less than two times the term value.

2 x 16

31

The sum of the first n terms in the series is equal to two times the term value of the n^{th} term, minus one

or

one less than two times the term value of the n^{th} term.

D57. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32 64

(Σ): 1 3 7 15 31 63

For n = 5, $\overset{n}{\Sigma}$ = ____

For n = 6, $\overset{n}{\Sigma}$ = ____

For n = 7, $\overset{n}{\Sigma}$ = ____

The rule for summing this number series is:

in symbols: $\overset{n}{\Sigma}$ = ______________________________

in words: The sum of the first n terms in this series is equal to ______________________________.

D58. (n): 1 2 3 4 5 6 7 8

(T): 1 2 4 8 16 32 __ __

(Σ): 1 3 7 15

Rule: We had the rule $\overset{n}{\Sigma} = T_{n+1} - 1$ for summing this series before. Now we have discovered another rule which is

in symbols: $\overset{n}{\Sigma} = 2T_n - 1$

in words: The sum of the first n terms in the series is equal to 2 times the term value, minus one. Or in other words, $\overset{n}{\Sigma}$ = one less than two times the term value.

Examples:

For n = 3, $\overset{n}{\Sigma} = (2 \times 4) - 1$ or $\overset{n}{\Sigma} = 7$

For n = 4, $\overset{n}{\Sigma} = (2 \times 8) - 1$ or $\overset{n}{\Sigma} = 15$

For n = 5, $\overset{n}{\Sigma} = (\quad\quad) - 1$ or $\overset{n}{\Sigma}$ = __

Copy the rule here:

in symbols: $\overset{n}{\Sigma}$ = ______________________________

in words: The sum of the first n terms in this series is equal to ______________________________

Answers

$2T_n - 1$

$T_{n+1} - 1$

D59.

$$\begin{tikzcd}[row sep=small, column sep=small]
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
(T): & 1 & 2 & 4 & 8 & 16 & 32 \\
(\Sigma): & 1 & 3 \arrow[ur] & 7 & 15 \arrow[u, dashed]
\end{tikzcd}$$

This series may be summed with at least two rules: One that uses T_n and one that uses T_{n+1}. Write both of the rules in symbols.

$\overset{n}{\Sigma}$ = ____________________ (Rule using T_n)

$\overset{n}{\Sigma}$ = ____________________ (Rule using T_{n+1})

one

127

D60. Now here is a slightly more complex series. First let's just look at the series itself. Can you see how it was constructed?

Series: 1 3 7 15 31 63 __ __

It's almost as if each term were multiplied by two, but that isn't quite right. Actually, each term is the preceding term multiplied by two, plus _____. Check this out.

What is the term after 63? _____

$n + 1$

T_{n+1} and $n + 1$

D61. Now that you are sure what the series is, we can write it down the usual way.

$$\begin{tikzcd}[row sep=small, column sep=small]
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_\_ & \_\_
\end{tikzcd}$$

Look at the arrows. They mean we are looking for a relation between $\overset{n}{\Sigma}$ and T_{n+1} or between $\overset{n}{\Sigma}$ and _____ or between $\overset{n}{\Sigma}$ and both _____ and _____.

31

D62.

$$\begin{tikzcd}[row sep=small, column sep=small]
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_\_ & \_\_
\end{tikzcd}$$

Let's try a relation with T_{n+1} first. Can you see one?

When $\overset{n}{\Sigma} = 4$, $T_{n+1} = 7$

When $\overset{n}{\Sigma} = 11$, $T_{n+1} = 15$

When $\overset{n}{\Sigma} = 26$, $T_{n+1} =$ __

Answers

- (minus) 4

- 5

D63.

$$\begin{array}{lccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (T): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ \\ & & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\ (\Sigma): & 1 & 4 & 11 & 26 & 57 & \_\_ & \_\_ \end{array}$$

When $\overset{n}{\Sigma} = 4$, $T_{n+1} = 7$

When $\overset{n}{\Sigma} = 11$, $T_{n+1} = 15$

When $\overset{n}{\Sigma} = 26$, $T_{n+1} = 31$

Can we get from T_{n+1} to $\overset{n}{\Sigma}$ by addition? No. Try subtraction.

When $T_{n+1} = 7$, $\overset{n}{\Sigma} = 4$, or $T_{n+1} - 3$

When $T_{n+1} = 15$, $\overset{n}{\Sigma} = 11$ or T_{n+1} —

When $T_{n+1} = 31$, $\overset{n}{\Sigma} = 26$ or T_{n+1} —

6

D64.

$$\begin{array}{lccccccc} (n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (T): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ \\ & & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\ (\Sigma): & 1 & 4 & 11 & 26 & 57 & \_\_ & \_\_ \end{array}$$

When we have $T_{n+1} = 7$, we can get $\overset{n}{\Sigma}$ by subtracting 3.

When we have $T_{n+1} = 15$, we can get $\overset{n}{\Sigma}$ by subtracting 4.

When we have $T_{n+1} = 31$, we can get $\overset{n}{\Sigma}$ by subtracting 5.

When we have $T_{n+1} = 63$, we can get $\overset{n}{\Sigma}$ by subtracting __.

Answers

from n + 1

n + 1 or 6

D65.

$$\begin{tikzcd}[column sep=small, row sep=small]
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_\_ & \_\_
\end{tikzcd}$$

Something looks regular about this.

When $T_{n+1} = 7$, then $\overset{n}{\Sigma} = 7 - 3$

When $T_{n+1} = 15$, then $\overset{n}{\Sigma} = 15 - 4$

When $T_{n+1} = 31$, then $\overset{n}{\Sigma} = 31 - 5$

Where can we get these numbers to subtract from T_{n+1}? ________________

For example, when $T_{n+1} = 63$, what do we subtract? ____________

Yes (at least we hope it did)

D66.

$$\begin{tikzcd}[column sep=small, row sep=small]
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_\_ & \_\_
\end{tikzcd}$$

When $T_{n+1} = 7$, then $\overset{n}{\Sigma} = 7 - 3$. When $T_{n+1} = 15$, then $\overset{n}{\Sigma} = 15 - 4$, etc.

What we are subtracting in each case is n + 1, isn't it? Look at T_{n+1} and then look at the number in the row above it(that is, at n + 1). Does this check out? _______

$\overset{n}{\Sigma} = T_{n+1} - (n+1)$

The sum of the first n terms in the series is equal to the value of the $n + 1^{th}$ term, minus n + 1.

57

120

120

D67.

$$\begin{tikzcd}[column sep=small, row sep=small]
(n): & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(T): & 1 & 3 & 7 & 15 & 31 & 63 & \_\_ \\
(\Sigma): & 1 \arrow[ur] & 4 \arrow[ur] & 11 \arrow[ur] & 26 \arrow[ur] & 57 \arrow[ur] & \_\_ & \_\_
\end{tikzcd}$$

Are you ready now to state a general rule? A rule for summing this series is

in symbols: $\overset{n}{\Sigma} =$ _____

in words: The sum of the first n terms in the series is equal to ____________________________

Examples: (Use the rule)

For n = 5, $\overset{n}{\Sigma} =$ _____

For n = 6, $\overset{n}{\Sigma} =$ _____

When $T_{n+1} = 127$, $\overset{n}{\Sigma} =$ _____

Answers	
	D69. Now let's review.
$T_{n+1} - 1$	(n): 1 2 3 4 5
$2T_n - 1$	(T): 1 2 4 8 __
31	(Σ): 1 3 7 15 __
16	Write two different rules in symbols.
	$\overset{n}{\Sigma}$ = __________
	$\overset{n}{\Sigma}$ = __________
	Complete the example.
	for n = 5, $\overset{n}{\Sigma}$ = _____
	for n = 5, T_n = _____
n^2 or n x n	(n): 1 2 3 4 5
25	(T): 1 3 5 7 __
9	(Σ): 1 4 9 16 __
	Write the rule in symbols.
	$\overset{n}{\Sigma}$ = __________
	Complete the example
	for n = 5, $\overset{n}{\Sigma}$ = _____
	for n = 5, T_n = _____
$T_{n+1} - (n + 1)$	(n): 1 2 3 4 5
57	(T): 2 3 7 15 __
31	(Σ): 1 4 11 26 __
	Write the rule in symbols.
You have completed the program! Congratulations!	$\overset{n}{\Sigma}$ = __________
	Complete the example
	for n = 5, $\overset{n}{\Sigma}$ = _____
	for n = 5, T_n = _____

TRANSFER TEST

Given immediately following completion of learning program and a-ain after a four weeks interval.

PROBLEM ST-1

(n): 1 2 3 4 5 6 7

(T): 1 3 9 27 81 243

(Σ): 1 4 13 40 121

Write the rule in symbols.

$\overset{n}{\Sigma}$ = ____________

Problem ST-1 Hint One

(n): 1 2 3 4 5 6 7

(T): 1 3 9 27 81 243

(Σ): 1 4 13 20 121

What must be done to the number 9 to get the number 4?

Problem ST-1 Hint Two

(n): 1 2 3 4 5 6 7

(T): 1 3 9 27 81 243

(Σ): 1 4 13 40 121

The number 4 is half of one less than 9

The number 13 is half of one less than 27

PROBLEM ST-2

(n): 1 2 3 4 5 6

(T): 1/3 2/3 1 1/3 2 2/3 5 1/3 10 2/3

(Σ): 1/3 1 2 1/3 5 10 1/3 21

Write the rule in symbols.

$\overset{n}{\Sigma}$ = ____________

Problem ST-2 Hint One

(n):	1	2	3	4	5	6
(T):	1/3	2/3	1 1/3	2 2/3	5 1/3	10 2/3
(Σ):	1/2	1	2 1/3	5	10 1/2	21

What must be done to get 1 from 1 1/3?

Problem ST-2 Hint Two

(n):	1	2	3	4	5	6
(T):	1/3	2/3	1 1/3	2 2/3	5 1/3	10 2/3
(Σ):	1/3	1	2 1/3	5	10 1/3	21

The number 1 is one-third less than 1 1/3.

The number 2 1/3 is one-third less than 2 2/3.

PROBLEM ST-3

(n):	1	2	3	4	5
(T):	1	7	19	37	61
(Σ):	1	8	27	64	125

Write the rule in symbols.

$\overset{n}{\Sigma}$ = ____________

Problem ST-3 Hint One

(n):	1	2	3	4	5
(T):	1	7	19	37	61
(Σ):	1	8	27	64	125

What must be done to the number 2 to get 8?

Problem ST-3 Hint Two

(n):	1	2	3	4	5
(T):	1	7	19	37	61
(Σ):	1	8	27	64	125

The number 8 is 2 x 2 x 2.

The number 27 is 3 x 3 x 3.

PROBLEM ST-4

(n):	1	2	3	4	5	6
(T):	1	4	10	22	46	94
(Σ):	1	5	15	37	83	177

Write the rule in symbols.

$\overset{n}{\Sigma}$ = ___________

Problem ST-4 Hint One

(n):	1	2	3	4	5	6
(T):	1	4	10	22	46	94
(Σ):	1	5	15	37	83	177

What must be done to 4 to get 1?

Problem ST-4 Hint Two

(n):	1	2	3	4	5	6
(T):	1	4	10	22	46	94
(Σ):	1	5	15	37	83	177

The number 1 is 4 minus (2 + 1).

The number 5 is 10 minus (3 + 2).

RETENTION TEST

Problems from the learning program given as a retention measure after a four week interval.

Problem P-1, 2

(n): 1 2 3 4 5

(T): 1 2 4 8

(Σ): 1 3 7 15

Write two different rules in symbols.

$\overset{n}{\Sigma} =$ ____

$\overset{n}{\Sigma} =$ ____

Problem P-3

(n): 1 2 3 4 5

(T): 1 3 5 7

(Σ): 1 4 9 16

Write the rule in symbols.

$\overset{n}{\Sigma} =$ ____

Problem P-4

(n): 1 2 3 4 5

(T): 1 3 7 15

(Σ): 1 4 11 26

Write the rule in symbols.

$\overset{n}{\Sigma} =$ ____